A Rationale for the Existence of Sovereign Lending Mechanisms

Roberto Pancrazi*　　Luca Zavalloni †

University of Warwick　　Central Bank of Ireland

September 17, 2018

PRELIMINARY AND INCOMPLETE

Abstract

We propose a rationale for the existence of lending mechanisms, such as the IMF and the ESM, which provide lending to distress sovereigns. First, we show that perfectly competitive markets for sovereign bonds are characterized by an externality: when the ownership of debt is anonymous and dispersed, the market price of newly issued bond might be too low to avoid default, even though preventing default would be in the interest of existing creditors. Then, we show that a policy maker/institution can address this externality by facilitating lending to the SOE government at more favourable terms. This policy is ex-post Pareto improving: the borrower can enjoy credit at lower interest rates, while investors gain from the delay in default even though they are directly financing the policy. Finally, the ex-ante gains are tightly related to the fiscal policy used to finance the intervention. We prove the existence of a Pareto set of fiscal policies that makes the intervention beneficial for all agents.

Keywords: Lending mechanism, Sovereign bond market, Endogenous Default Model.

JEL Classification: E44, E58.

*University of Warwick, Economics Department, Coventry CV4 7AL, United Kingdom; E-mail address: R.Pancrazi@warwick.ac.uk

†Central Bank of Ireland, North Wall Quay Dublin 1, Ireland; E-mail address: Luca.Zavalloni@centralbank.ie

1 Introduction

The recent European sovereign crisis has called for unconventional and unprecedented policy responses to overcome a long period of fiscal distress, financial turmoil, and uncertainty. One of the main innovation has been the creation of financial assistance programmes for member states of the eurozone in financial distress, such as the European Financial Stability Facility (EFSF) and the European Financial Stabilisation Mechanism (EFSM). These temporary programs were replaced in September 2012 by the creation of an institutionalized intergovernmental organization, the European Stability Mechanism (ESM). In short, the ESM provides lending at favourable rates to the Eurozone member countries, conditional on them first signing a *Memorandum of Understanding*.

In this paper we rationalize the existence of these lending mechanisms; the starting point is highlighting that perfectly competitive markets for sovereign bonds that carry default risk are characterized by an externality. In fact, in a competitive market, in which the ownership of debt is anonymous and dispersed, the market price of newly issued bonds might be too low to avoid default, even though preventing default would be in the interest of existing creditors. Consider a country at the verge of default; if it issues new bonds, their market price would intrinsically be low. However, it is also the fact that the country needs to borrow at an high interest rate that, in turn, affects the default decision. This prospect is unwelcome by existing bondholders since they would like, if they could, to offer better borrowing conditions to the troubled economy in order to delay default. In a competitive market, however, existing bondholders are not able to affect the bond price. This gives rise to what we label as the *non-exclusivity externality*. This externality arises because of the existence of a large number of atomistic lenders: at the endogenous debt limit, the expected present value of returns to new lenders becomes negative, even though the value for existing lenders remains positive, and additional loans are profitable only because they improve previous loans. However, if there were only one lender in the market, this externality would disappear, as that lender would be able to extend new loans at conditions that nobody else would be willing to accept.[1]

[1] In fact, as pointed out by Hellwig (1977) in a more general setting, credit granted to a single borrower is not an homogeneous good, since later loans affect the return on earlier loans. Therefore,

An important consequence of this externality is that the endogenous debt limit for the sovereign economy that borrows in competitive markets is much lower than the one implied by eliminating the externality.

There are three contributions in our paper. First, we highlight the presence of an externality in a standard incomplete market model for sovereign bonds. Second, we define a policy intervention that addresses this externality and we highlight its ex-post equilibrium properties; our policy is closely related to the goals and practice of the European Stability Mechanism and the International Monetary Fund. Third, we investigate the ex-ante properties of an equilibrium that internalizes a possible future intervention. The main results can be summarized as follow. Ex-post the intervention of the lending mechanism is Pareto improving; the borrower can enjoy credit at lower interest rates, while investors gain from the delay in default even though they are directly financing the policy. Ex-ante, the welfare properties of the policy crucially depend on the fiscal policy used to finance the intervention. Nevertheless, we show that there is a set of fiscal policies that obtain an ex-ante Pareto improvement.

Given the relevance of the European sovereign crisis, the recent literature has investigate possible causes of market failures and explored possible policy solutions. One strand of the literature has rationalized the scope of intervention policies in the bond market with preventing self-fulfilling crises. This line of research has been pioneered by Diamond and Dybvig (1983) and includes works by Gertler and Kiyotaki (2015), Lorenzoni and Werning (2014), Broner et al. (2014), Corsetti et al. (2016), De Grauwe and Ji (2013). In a nutshell, the basic idea is that the economy can be stuck in a bad equilibrium caused by investors' pessimistic expectations and policy intervention is able to revert the economy to a good equilibrium by acting as a lender of the last resort. Whereas most of this literature investigate the effects of unconventional policies such as the Outright Monetary Transactions, we focus on a less explored issued, that is the merits of lending mechanism, by offering a novel rationale for their existence.[2] Another strand of the literature has explored the degree of inefficiency created by the debt di-

such an existing creditor is able to extend new loans at conditions that nobody else would be willing to accept.

[2]Examples of work that evaluate the effects of various ECB policies are Merler et al. (2012), De Pooter et al. (2012), Eser and Schwaab (2012), Altavilla et al. (2014), Krishnamurthy et al. (2015), Falagiarda and Reitz (2015), and Szczerbowicz et al. (2015), among others.

lution problem; as discussed in Bolton and Jeanne (2009), Chatterjee and Eyigungor (2015), Hatchondo et al. (2016), the debt-dilution problem is caused by the government's lack of commitment not to decrease the value of debt issued in the past by issuing new debt. Let us first summarize our framework. There are three agents in the economy: a small open economy (SOE, henceforth) government, international investors, and a policy maker. *(i)* The small open economy starts in a recessionary state and finances its consumption streams issuing long-term bonds. Stochastically, the exogenous income eventually jumps to a good state, in which it will remain forever. However, if the recession is long lasting the government might accumulate too much debt and optimally decide to default. In this case, it stops interest repayments to bondholders and remains in financial autarky until the recession is over. Once the economy has recovered, the government re-enters the financial market and repays the investors after a renegotiation of the debt burden. Hence, the problem of the SOE is standard and it is in the same spirit of the endogenous default framework as in Eaton and Gersovitz (1981). *(ii)* Risk neutral international investors buy sovereign bonds in a perfectively competitive market and bear the risk of a loss due to the government's insolvency in case of default. Their optimal decision results in an equilibrium bond price. *(iii)* Finally, a policy maker might decide to implement a policy, described in details below, which offers to the SOE government the possibility to borrow at more favourable conditions, in order to extend interest repayment to bond holders and delay default. The intervention is budget-balanced and financed by levying taxes to investors.

The policy intervention in our framework works as follows. When the borrowing government requires financial assistance, the policy maker can address the *non-exclusivity externality* by subsidizing new lending to the small open economy. The size of the subsidy, and therefore the bond price after the policy implementation, is chosen so that the SOE is indifferent between defaulting and continuing to borrow at the policy price. As a consequence, the small open economy can borrow at better conditions than the one offered by competitive investors absent the policy. This way the government is willing to delay default and continue to service the debt. The policy is budget balanced and the cost of the policy is financed by taxes levied to investors. We show that this type of intervention is, at the moment of its implementation, Pareto improving. In fact, while the SOE government is kept indifferent by construction, investors are better off because the benefits obtained by the increased probability to receive a full repayment of their existing loans exceeds the expected loss from financing the intervention. In addition, we show that the policy intervention has always a limited duration. In fact, after the policy starts, if the recession continues to last, the authority needs to give always better and better condition to keep the SOE away from default. As a consequence, the tax burden for investors rises and eventually it is such that it perfectly counteracts the benefit of the intervention. At that point the intervention stops, and the SOE defaults.

The final part of the paper concerns welfare. Is the policy ex-ante welfare improving? The first step to answer this question is realizing that the ex-ante welfare of investors and of the SOE are a function of how the policy is financed. We assume that the tax to investors is composed by two components: a proportional and distortive tax per-unit of asset and a lump-sum tax, which taxes each investor independently of asset holding. A parameter, α, determines the proportion of the distortive component. Ex-ante, which means when default has not yet occurred but it is known the possibility that a policy intervention will take place, the welfare gain of the existence of the policy relates to the degree by which the market bond price does not internalize the overall cost of the intervention, which depends on α. In addition, we shows that the monetary/fiscal authority can use the fiscal policy α as redistribution instrument between investors' welfare and the SOE's welfare. Financing the intervention by imposing heavy proportional taxes to bond holding depresses bond prices and the country's welfare in favor of investors' welfare. On the contrary, by attenuating the dependence of the tax upon bond holding, the policymaker creates an over valuation of the bond price which diminishes investors' welfare in favor of the country's welfare by creating an implicit fiscal transfer to the latter. Finally, we prove the existence of a Pareto set of fiscal policy, α, for which the intervention is ex-ante Pareto improving.

The final comment concerns the essence of the policy intervention that addresses the *non-exclusivity externality*, which is fundamentally a transfer from existing creditors (who pay the taxes) to new investors that underwrite bonds on the primary market (who enjoy the subsidy). Whereas a subsidy on investment is a policy that is not controversial as it can be easily implemented using market instruments, the ability of the policymaker to tax bondholding could be disputed. Nevertheless, once the mechanism is clear, it is then easy to think to alternative arrangements that can lead to the same outcome. Similarly to our policy, we can think that a credit line is provided directly by an international institution that finances the intervention through taxes. Alternatively, we could think to more nuanced contractual arrangements. For example, we can think to a complex seniority structure that, after some debt level, gives progressively increasing seniority to new bondholders, or equivalently that debt is progressively restructured conditional on the country continuing to borrow from the market.

1.1 Related Literature

We have already mentioned that a contribution of our paper is to provide an additional rationale for policy interventions in the sovereign bond market, in addition to the equilibrium selection motive as in Gertler and Kiyotaki (2015), Lorenzoni and Werning (2014), Broner et al. (2014), Corsetti et al. (2016), De Grauwe and Ji (2013), and Diamond and Dybvig (1983), and the debt dilution problem as in Bolton and Jeanne (2009), Chatterjee and Eyigungor (2013), Hatchondo et al. (2016). The *non-exclusivity externality* we introduce in this paper is in the same spirit of Hellwig (1977), although in his framework debt limits are exogenous whereas in our setting they are endogenously determined by the optimal default decision.

The *non-exclusivity externality* can coexist with, but is independent from, the debt-dilution externality. Interestingly, the solution for solving the debt-dilution problem is rather different to the one that mitigates the externality in our paper. In fact, Hatchondo et al. (2016) show that a way to eliminate the debt-dilution problem features a tax on debt for the bond-issuing country that benefits long-term holders. In our framework, instead, we show that investors themselves are willing to finance a policy that eliminates the market failure, as it will become clear later.

Our paper is closely related to Hatchondo et al. (2014), which show that a sovereign model a la Eaton and Gersovitz (1981) can account for what they label "voluntary debt exchange", i.e. episodes in which both the government and its creditors are likely to benefit from reductions in the government's debt burden. There are two main differences with our paper. First, we show that even absent debt restructuring, the existence of bailout facilities that facilitate lending to the government at better borrowing conditions can be in the interest of both creditors and debtors and mitigate an externality proper of the competitive market for sovereign debt. Second, our papers focuses on a simple model where analytic results can be derived. In particular we show that, absent initiatives to coordinate creditors, the government defaults at a sub-optimally low level of debt and policies aimed at solve this externality may largely reduce spreads and extend borrowing limits.

It is also important to highlight what our paper and our results are silent about. Our welfare analysis considers only the direct effects of solving the *non-exclusivity externality*, while we ignore other indirect channels that possibly affect the overall welfare of the economy. For example, lower sovereign spreads may have a beneficial effect on the real economy, through the link between sovereign bonds and the balance-sheet of the banking sector, as highlighted in Gennaioli et al. (2014), and in Popov and Van Horen (2015), among others.

The rest of the paper is structured as follows. In section 2 we introduce a simplified two period model to highlight the mechanism driving the externality. In section 3 we outline the general environment in continuous time. In section 4 we describe the competitive equilibrium. In section 5 we describe the policy intervention and its ex-post properties. In section 6 we discuss the ex-ante equilibrium properties. In section 7 we characterize the ex-ante welfare implications of the policy. In section 8 we conclude with final remarks.

2 Two-period Model

Small Open Economy The economy lasts two periods: $t = 1, 2$. A representative agent (henceforth government) in a small open economy (SOE) issues non-contingent bonds to smooth her consumption. In period 1 the economy has low endowment, $y_1 = y_L$. In period 2 the endowment could be either high, $y_2 = y_H$, or low, $y_2 = y_L$, respectively with probability p and $1-p$. The country starts with a level of asset $B_1 < 0$, which means that the country has some initial debt. We assume that the government has a logarithm utility function $u(c) = \log(c)$, where c denotes consumption. The concavity of the utility function, together with the assumption that the period-2 income might be higher than the period-1 income, provides a motive for borrowing. The government can default on its debt in period 1 or in period 2 provided that endowment is low. However, we assume that the government cannot default if income is high. This assumption implies that there is a zero cost of default in the low state and in infinite cost of default in the high state; it aims to capture in a reduced form the fact that it might be too costly for the government to default in a boom, so that an high income realization effectively acts as a commitment technology not to default. We indicate with $\mathbb{1}_t^D$ the default decision at time t, where $\mathbb{1}_t^D = 1$ denotes default and $\mathbb{1}_t^D = 0$ denotes repayment. Default implies no penalty other than exclusion from financial markets.

Risk Neutral Investors As standard, we assume that atomistic foreign creditors have access to an international credit market in which they can borrow or lend as much as needed at a constant international interest rate, which we assume to be zero. They have perfect information regarding the economy's endowment process and can observe the level of income every period. Creditors are assumed to price defaultable bonds in a risk neutral manner such that in every bond contract offered they break even in expected value.

The problem for the government is:

$$
\begin{aligned}
& \max_{\{c_1, c_2, B_2, \mathbb{1}_1^D, \mathbb{1}_2^D\}} \log(c_1) + \mathbb{E}\log(c_2) \\
s.t. \quad & c_1 + qB_2 = y_L + (1 - \mathbb{1}_1^D)B_1, \\
& B_2 = 0 \quad \text{if} \quad \mathbb{1}_1^D = 1, \\
& c_2 = \begin{cases} y_H + B_2, & prob = p \\ y_L + (1 - \mathbb{1}_2^D)B_2, & prob = 1 - p. \end{cases} \\
& B_1 < 0, \text{given.}
\end{aligned}
\tag{1}
$$

Some remarks are in order. First, without loss of generality we have assumed that there is no discounting from period 1 to period 2. Second, q denotes the bond price. Third, defaulting in period 1, i.e. $\mathbb{1}_1^D = 1$ implies that the government does not repay its initial debt, B_1, and that it is excluded from the financial market so that in that case necessarily $B_2 = 0$.

2.1 Competitive Equilibrium

Definition 1. A Competitive Equilibrium for this economy is defined as a set of policies for consumption in period 1, c_1, and in period 2, c_2, for government's asset holdings B_2, a default decision in period 1 and period 2, $\mathbb{1}_1^D$ and $\mathbb{1}_2^D$, and a bond price q, such that:

1. Taking as given the bond price q, the government's asset holdings and default decisions satisfy the government optimization problem.

2. The bond price, q, being consistent with creditors' expected zero profits, reflects the government's period-2 default probability.

3. Taking as given the government policies, consumption satisfies the resource constraint.

To characterize the competitive equilibrium, we solve the model by backward induction. Notice that defaulting in period 2, i.e. $\mathbb{1}_2^D = 1$, simply implies that the government will not repay its debt and no further penalties occur. As a result, the government will always default in the low income state; formally, $y_2 = y_L \implies 1_2^D = 1$. Using this result, we can state the value of not defaulting in period 1, $W_1^{ND}(B_1, q)$, for a generic bond price q, as:

$$W_1^{ND}(B_1, q) = \max_{B_2} \log(y_L + B_1 - qB_2) + (1-p)\log(y_L) + p\log(y_H + B_2). \tag{2}$$

Solving for B_2, gives the optimal asset/debt position:

$$B_2^*(B_1, q) = \frac{\frac{p}{q}(y_L + B_1) - y_H}{1+p}. \tag{3}$$

If instead the government defaults in period 1, its value, W_1^D is:

$$W_1^D = \log(y_L) + (1-p)\log(y_L) + p\log(y_H).$$

In a competitive market the bond price is equal to the period-2 probability of repayment, which is the probability of a high income realization; therefore, $q = p$. The government will optimally default whenever $W_1^D \geq W_1^{ND}(B_1, p)$. We can easily show that there exists a threshold $\bar{B}_1$ such that $W_1^{ND}(\bar{B}_1, p) = W_1^D$. If B_1 is above that threshold the government does not default in period 1, while if B_1 is below the threshold, the government defaults in period 1. The result is formalized by the following proposition.

Proposition 1. *In a competitive equilibrium* $\exists! \bar{B}_1 < 0$ *such that:* $B_1 \leq \bar{B}_1 \iff \mathbb{1}_1(B_1, p) = 1$.

See Appendix A.1 for the proof.

2.2 Non-Exclusivity Externality

We now show that the competitive market is characterized by an externality. Assume that the initial level of debt is $\bar{B}_1$, and the government, being at the indifference point between defaulting and not defaulting, opts to default in period 1. In this case, existing lenders are going to lose their investment. Also notice that, since debt is non-exclusive and lenders are atomistic, they are not willing to underwrite any positive amount of bond at a price higher then q; therefore, $q = p$. However, they do not internalize that their behavior generates a price that incentives the government to default. In fact, the fact that the country can only borrow at a price $q = p$ is also the reason why the country will default in period-1. Everything else equal, better borrowing conditions would avoid default. In this section we show: (i) that agents in the economy would all be better off if the bond price was slightly higher than the market price; and (ii) how to simply implement that price.

First, notice that, by the envelope theorem, the government's value of non-defaulting in equation (2) is a positive function of the bond price q: $B_1 < 0 \Rightarrow \frac{\partial W_1^{ND}(B_1, q)}{\partial q} > 0$.

Therefore, for any price $p + \xi$, with $\xi > 0$, the government would not default at $\bar{B}_1$ and would optimally borrow the quantity $\tilde{B}_2 = \frac{\frac{p}{p+\xi}(y_L + \bar{B}_1) - y_H}{1+p}$.[3]

The competitive market, per-se, cannot support a price higher than p. However, a higher price can be implemented by introducing a simple subsidy ξ per unit of bonds

[3]Notice that $\tilde{B}_2 < 0$ only when $\xi < \frac{p(y_H - y_L - B_1)}{y_H}$, which puts an upper bound on ξ.

underwritten on the primary market, which we assume is financed by taxing existing bondholders. This policy can be interpreted as a stylized version of the goals of lending mechanism: they provide lending at a favorable rate, $p+\xi$, and they finance this policy with the contribution of the member countries.

Let us first analyze how the subsidy affects existing investors' welfare. Under the competitive price $q = p$, existing bondholders will lose all their investment, and therefore their payoff is equal to zero. Under the alternative price $p + \xi$, they will get back their investment $-B_1 > 0$ and they will pay the cost of the transfer, which is equal to the unit cost of the subsidy, ξ, times the total amount of new bond optimally sold by the country and acquired by new investors, equal to $\tilde{B}_2$. Given that they are risk neutral, their welfare gain from the policy is:

$$V^{oldI}(p+\xi) - V^{oldI}(p) = \begin{cases} -B_1 + \xi\tilde{B}_2 & \text{if} \quad \xi > 0 \\ 0 & \text{if} \quad \xi \leq 0, \end{cases}$$

where $V^{oldI}(q)$ denotes the welfare of existing investors when the bond price equals to q. The fact that default is a binary choice generates a discontinuity in the payoff to old investors.

Now let us look at new investors. First recall that the economy will default surely if in period-2 the realization of income is low. Hence, if new investors buy the bond price at $\tilde{q} = p+\xi$, they would make an ex-ante loss. However, the gross subsidy they receive, equal to $\xi(-\tilde{B}_2)$ implies that new investors break even in expectations when the bond price is $p + \xi$. Their welfare gain from the policy is:[4]

$$V^{newI}(p+\xi) - V^{newI}(p) = -\xi(-\tilde{B}_2) + \xi(-\tilde{B}_2) - 0 = 0.$$

This implies that we can always define a ξ sufficiently small such that the policy makes both existing bondholders and the government better off, as shown by the following Proposition.

Proposition 2. *The competitive equilibrium is suboptimal. There exists $\bar{\xi}$ such that for*

[4] Recall that $V^{newI}(p) = 0$, since at the bond price p the SOE defaults in period 1 and therefore investors do not acquire any asset.

$\xi \in (0, \bar{\xi}]$ *a Pareto improvement over the competitive equilibrium can be obtained.*

See Appendix A.2 for the proof.

Figure 1 shows the welfare of existing investors as a function of ξ. At the competitive price, $q = p$, and therefore $\xi = 0$, existing investors make a loss since the country defaults. A price higher then p create s a jump on welfare since the country will not default and investors will be repaid. Then, the higher is the unity subsidy, the higher is the cost for investors, since not only the unit subsidy obviously increases, but also the amount of issued bond B_2 increases, since the SOE will optimally demand more debt when borrowing condition improve. Importantly, our result shows that small deviations from the competitive price $q = p$ are Pareto efficient since both the SOE and old investors are better off, while new investors are indifferent.

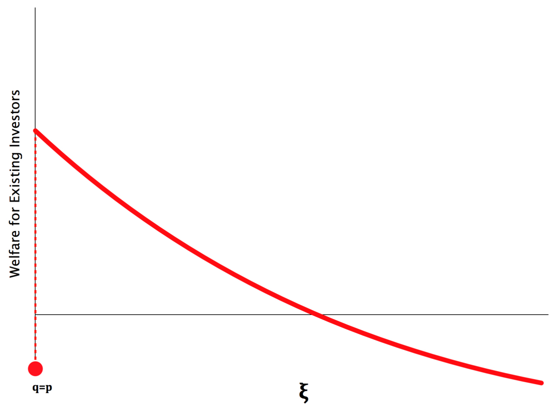


Figure 1 – Existing investors' welfare as a function of the subsidy ξ

This two-period model shows the heart of the non-exclusivity externality: the competitive market cannot price the incentives of old investors to avoid default and that results in a loss in efficiency. However, this simple model cannot answer interesting questions related to the proposed solution for the externality, such as whether the policy is effective over longer horizons, whether default will be always avoided under such a policy, and what are the ex-ante effects of the policy once investors are aware of it. In order to answer these questions we now introduce a more complete infinite horizon model in continuous time.

3 Continuous time model

The Peripheral Economy and Uncertainty. A representative agent (henceforth government) in a small open economy issues non-contingent bonds to smooth her consumption. The consumption smoothing desire is motivated by the uncertainty about the exogenous income that the government is facing and that is the only source of uncertainty in the economy. We assume that time is continuous and the income process Y_t follows a continuous-time Markov chain.[5]

More specifically, we assume a two-state process, i.e. $\mathcal{Y} = \{y_L, y_H\}$: here y_L denotes a bad state in which income is low and $y_H > y_L$ denotes a good state in which income y_H is high. In the initial period, $t = 0$, the economy is in the bad state (recession): the government is poor and needs to borrow to finance consumption and satisfy coupon payments. Eventually the country recovers and jumps to the good state. However, the time at which the country exits the recession is uncertain. Once the country recovers, uncertainty is resolved and the country will remain in the good state forever after. These assumptions impose restrictions on the *infinitesimal generator* matrix that governs the transition of the process. It can be shown that the transition probability matrix in this case is:

$$P(t) = \begin{pmatrix} e^{-\lambda t} & 1 - e^{-\lambda t} \\ 0 & 1 \end{pmatrix},$$

with initial condition $y(0) = y_L$ and where the entry $P_{ij}(t)$ denotes the instantaneous probability to jump from state i to state j, at time t. The key remark is that the time of the jump from the low to high income, which we define as T^j, has exponential distribution with parameter λ.

We have, therefore, a two-stage game. In stage-1, the prospect of an income increase provides a motive for borrowing. Uncertainty is then fully resolved at some random date at which point we enter stage-2 and the government receives a constant stream of income y_H.

Remark. This setting, which is consistent with Hellwig (1977), allows us to derive ana-

[5]See Appendix A.3 for a formal definition of a continuous-time Markov chain.

lytic results. Assuming an absorbing high income state is not a key limitation, since we are, anyway, mainly interested in the dynamics during the low income state, which are obviously the main drivers of sovereign crisis and monetary intervention.

Asset Structure. The government issues non-contingent bonds. These bonds have coupons that decrease at a continuum rate δ. Hence, a bond issued at t promises to pay the sequence of coupons:

$$ke^{-\delta(s-t)}, \quad \forall s \geq t,$$

where $\delta \in (0,1)$ and $k > 0$. We normalize and set $k = \delta + r$, so that the bond price is equal 1 when the risk of default is zero at all future dates, and where r is the assumed risk-free rate in the economy. This well-known formulation of long term bonds is useful because it avoids having to carry the entire distribution of bonds of different maturities (see Hatchondo and Martinez (2009)). A bond issued at $t-j$ is equivalent to $e^{-\delta(t-j)}$ bonds issued at t, so the vector of outstanding bonds can be summarized by a single state variable b_t, which is equal to total debt in terms of equivalent newly issued bonds. The parameter δ controls the maturity of debt, with $\delta = 1$ corresponding to the case of a one period bond and $\delta = 0$ corresponding to the case of a consol.

Government and Default. We allow for the government to endogenously default on its debt obligations. A key simplifying assumption for our analysis is that default can occur only when income is in the low state, as justified in the previous section. Hence, by assumption we rule out default when the economy exits the recession phase. As it will be clear throughout the paper, we will focus our analysis mainly in the recession state, since it is arguably the time in which policy intervention is meaningful; hence, we believe that simplifying the dynamics of the model in the high income state does not bear a large cost.[6] We assume the following sequence of events: if the government defaults,

[6]This assumption, which allows us to derive analytical results, is in line with the vast empirical literature on sovereign defaults that links default episodes to periods of recession. Using quarterly data for 39 developing countries over the 1970-2005 period, Yeyati and Panizza (2011) show that defaults are associated with deep recessions; Tomz and Wright (2007) analyse defaults in a longer sample,1820-2004, and, although they find evidence that defaults also happens without severe recessions, the maximum default frequency occurs when output is at least 7 percent below trend. Hence, we believe that assuming no defaults in the good state of the economy is quite realistic. One can relax this assumption by assuming

which always happens when income is still in the low state y_L, the government stops any coupon repayment and the country is excluded from the financial market so that the economy lives in autarky. When the recession is over, which means when income jumps to the higher state y_H, the government renegotiates debt payments by repaying only a fraction $\phi \in [0, 1]$ of outstanding debt at default, and it gains back access to financial markets.

We denote with T the time of default. In the next section we characterize the choice of the optimal time of default; here we describe the constraints the government faces. There are two cases, then: (i) either the country jumps out of the recession at a time, T^j, which occurs after the time of default, T and the country defaults on its debt; (ii) or default never happens. In the first case, $T < T^j$, the government budget constraint is:

$$\begin{aligned} c(t) + q(t)\left(\dot{b}(t) + \delta b(t)\right) &= y_L + (r+\delta)b(t), \quad && \text{for } t < T^j \\ c(t) &= y_L, && \text{for } T \leq t < T^j, \\ c(t) + q(t)\left(\dot{b}(t) + \delta b(t)\right) &= y_H + (r+\delta)b(t), && \text{for } t \geq T^j \text{ and with } b(T^j) = \phi b(T). \end{aligned}$$

The first equation states the resource constraint prior to the default. $c(t)$ denotes consumption at time t, $b(t)$ denotes asset holding, $\dot{b}(t)$ denotes the instantaneous change in asset position, $q(t)$ is the bond price. The second equation indicates that the government is excluded from the financial market from the time of default, T, to the time in which it enters in the good economic state, T^j. The third equation describes the budget constraint from the time of the jump onwards. Two things are worth noticing; first, when the economy regains access to the financial market it starts with a renegotiated level of debt, $b(T^j) = \phi b(T)$;[7] second, since by assumption after the jump no default will occur, then $q(t) = 1$, $\forall t \geq T^j$, because we have normalized the price of a risk-free bond to unity.

In the second case, $T > T^j$, there is no default and the government budget constraint

an output cost of default and a risk of returning to the low state of the economy after the jump to the high state. This setting would be more similar to standard business cycle endogenous default models as in Arellano (2008), which requires numerical solutions.

[7]Recall that in our notation $b(t)$ denotes asset level, so that at debt is negative asset holding.

is:

$$c(t) + q(t)\left[\dot{b}(t) + \delta b(t)\right] = y_L + (r+\delta)b(t), \quad \text{for } t < T^j$$
$$c(t) + q(t)\left[\dot{b}(t) + \delta b(t)\right] = y_H + (r+\delta)b(t), \quad \text{for } t \geq T^j.$$

Investors. The economy is populated by a continuum of mass 1 of risk neutral atomistic and homogenous investors, which operate in a competitive financial market, and, therefore, take the bond price as given. Let $a(t)$ denote each investor's individual bond holdings, which, in our economy, are the counterpart of governments' bond, so that in equilibrium we will have that $a(t) = -b(t)$. Denote with $V(\cdot)$ the investor's lifetime utility from its trading activity. Then, the investor's problem at any time t before the jump and before default, i.e. $\forall t < \min\{T, T^j\}$, is:

$$V(a(t)) = \max_{\{a(s)\}_{s=t}^{T}} \int_t^T \left(-q(s)(\dot{a}(s) + \delta a(s)) + (r+\delta+\lambda)a(s)\right) e^{-(r+\lambda)(s-t)} ds + \\ + V(a(T))\, e^{-(r+\lambda)(T-t)}. \tag{4}$$

The integral captures the value of an investor's asset position throughout the uncertain times in which the economy is in a recession and the government might default at time T; in this time interval, the investor can increase her asset holding position at the price $q(s)$, with $t \leq s \leq T$, and this investment returns the coupon repayment, $r+\delta$, as well some capital gain in case the economy jumps in the higher income state, event with arrival rate λ. Notice, in fact, that the assumption about the income process makes y_H an absorbing state and, therefore, once the income has jumped in that state the government will never default, and, therefore, $q(t) = 1$, $\forall t \geq T^j$. On the contrary, default risk while income is low, implies that $q(t) \leq 1$, $\forall t \leq T$. Finally, the last term captures the value of the bond portfolio in case the government defaults, which includes future repayments when the economy exits financial autarky and renegotiates the debt payments. The observation that the absence of arbitrage opportunities implies that the value of asset holdings should be linear in the bond price, i.e. $V(a(t)) = q(t)a(t)$, leads to some straightforward results:

Proposition 3. *Bond Price. The bond price $q(t)$ satisfies the following conditions:*

1. *For any period before default/jump, that is $\forall t < \min\{T, T^j\}$, the law of motion of*

the price $q(t)$ satisfies:

$$\dot{q}(t) = (r + \delta + \lambda)(q(t) - 1);$$

2. *The bond price at default, $q(T)$, is:*

$$q(T) = \frac{\phi\lambda}{r + \lambda}.$$

   This price is also the market price in any period between default, T, and the time of the jump to the high income state, T^j.

3. *For any period after the jump, that is $\forall t \geq T^j$, the law of motion of the price $q(t)$ satisfies:*

$$q(t) = 1.$$

See Appendix A.4 for the proof. Condition 3 follows directly from the assumption that the government cannot default in the high income state. Hence, after the jump to the high income state occurs, the sovereign bond is equivalent to a risk-free asset. Condition 2 relates the price of the bond at an instant prior to default directly to the recovery rate of the bond, ϕ, and to the value of the bond net of the expected foregone interests prior to renegotiation. Condition 1 is the non-arbitrage condition derived for risk-neutral investors acting in a competitive market. Conditions 1 and 2 are at the heart of the *non-exclusivity externality*. In a competitive market investors are price takers, they are willing to underwrite a new bond only if the price is lower or equal to the present value of future repayments on that specific bond. A new bond can never be sold at a higher price even though that higher price may increase the value of existing bonds. That is because it is individually rational for each existing bondholder, being atomistic and anonymous, to shun the new issuance trying to free ride on the increase in value of the bonds already in its portfolio. Essentially it is as if bonds are priced by new investors at each point in time. As it will be clear later, this feature, when combined with market incompleteness generates an externality that makes the government default at an inefficiently too low level of debt.

4 Competitive Equilibrium

In this section we characterize the equilibrium path of debt, bond price, and default time resulting from the government's optimization problem. In the following we make the simplifying assumption that the international interest rate r is equal to the household discount factor ρ.

Government problem prior to default. We first specify the problem of the government, its value, and its default decision, when it faces a low income and has not yet defaulted. The government takes the path of the bond price as given and it chooses optimally the path for consumption $\{c(s)\}_{s=0}^{T}$ and the optimal time of default T, as follows:

$$W(b(0)) = \max_{\{c(t)\}_{t=0}^{T},T} \int_0^T e^{-(\rho+\lambda)t} \left[\log(c(t)) + \lambda W^j(b(t))\right] dt + W^d(b(T))e^{-(\rho+\lambda)T}, \quad (5)$$

$$\text{s.t. } \dot{b}(t) = \frac{1}{q(t)}\left(y_L - c(t) + (\rho+\delta)b(t)\right) - \delta b(t), \quad (6)$$

$$\dot{q}(t) = (\rho+\delta+\lambda)(q(t)-1), \quad (7)$$

$$q(T) = \frac{\lambda\phi}{\lambda+\rho}, \quad (8)$$

$$b(0) \text{ given,}$$

where $W^j(\cdot)$ is the value at the moment of the jump to the good income realization, and $W^d(b(T))$ is the value at default. The two values can be computed easily in closed form and their derivation is described in Appendix A.5.

The first constraint is the resource constraint of the government. The second constraint is the evolution of the bond price that follows from the investors' problem. The third constraint is the equilibrium bond price at time of default. Taking first order conditions, the continuous time Euler equation that characterizes the optimum is:

$$\frac{\dot{c}(t)}{c(t)} = \frac{\lambda}{q(t)}\left[c(t)W_b^j(b(t)) - 1\right], \quad \forall t \le T, \quad (9)$$

where $W_b^j(\cdot)$ denotes the derivative of the function $W^j(\cdot)$ with respect to b. See Appendix A.6 for the formal derivation.

Terminal conditions. The dynamic differential equation in (9), together with the differential equation for $\dot{b}(t)$ coming from the government resource constraint in (6) and the evolution of the bond market price for $\dot{q}(t)$ in (7), pin down the optimal path of consumption and asset holdings, given the terminal conditions for the three variables $c(T), b(T), q(T)$. The terminal condition for $q(T)$ follows directly from Proposition 3, while deriving the terminal conditions for $c(T)$ and $b(T)$ in the context of *free terminal time boundary value problems* is well established. A formal derivation is provided in Hartl and Sethi (1983) and applied in Hellwig (1977) in a similar context. In our case, these terminal conditions are pin down by the solution to the system:

$$\log(c(T)) - \log(y_L) = \lambda\left[W^j(\phi b(T)) - W^j(b(T))\right] - W_b^d(b(T))\dot{b}(T) \tag{10}$$

$$c(T) = y_H + \rho\phi b(T), \tag{11}$$

$$\dot{b}(T) = \frac{\rho+\lambda}{\lambda\phi}\left[y_L - c(T) + (\rho+\delta)b(T)\right] - \delta b(T). \tag{12}$$

See Appendix A.7 for the formal derivation

The first equation is the transversality condition and should be interpreted as a trade-off in the time dimension. The left hand side represents the benefit of delaying default of one instant, which stems from the possibility to consume more than in autarky. The right hand side represents the cost of delaying default of one instant, which is composed by two terms: (i) the foregone opportunity to default in case the jump occurs at that instant, which is a function of the arrival rate λ, and of the renegotiation parameter ϕ; (ii) the disutility to default with an higher debt burden. The second equation stems from the first order conditions and relates to the fact that, upon default, the marginal utility of issuing an additional unit of bond should be equal to the disutility from defaulting with an additional unit of debt. The third equation is the government budget constraint. The solution of this system of three equations in three unknown, $b(T)$, $c(T)$, $\dot{b}(T)$, determines these terminal values.

Competitive Equilibrium. We are now ready to define a competitive equilibrium for the economy, prior to the default or jump.

Definition 2. A Competitive equilibrium is a bond price sequence $\{q(t)\}_{t=0}^{T}$, a saving sequence $\{b(t)\}_{t=0}^{T}$, a consumption sequence $\{c(t)\}_{t=0}^{T}$, and an optimal default time T for the SOE government, and an asset holding sequence $\{a(t)\}_{t=0}^{T}$ for investors, such that, given $\{q(t)\}_{t=0}^{T}$:

(i) investors, solving the problem in (4), break-even in expectation.

(ii) the government solves the problem in (5)-(8).

(iii) the government defaults at T, if $T < T^j$.

(iv) bond markets clear, i.e. $b(t) = -a(t)$, $\forall t$.

Once again, for convenience, we focus on the equilibrium for any $t < \min\{T, T^j\}$, since this is the relevant case for which the policy intervention is meaningful. It is straightforward to define and derive the equilibrium condition under the alternative scenarios: the bond price after default or after the jump is described in Proposition 3, whereas the government budget constraints are described in Section 3. Since these cases are not relevant for the scope of the paper we omit their formal description.

Hence, the equilibrium before default is characterized by the following system of differential equations:

$$
\begin{aligned}
\dot{q}(t) &= (\rho + \delta + \lambda)(q(t) - 1), \quad \forall t \leq T \\
q(t)\dot{b}(t) &= y_L - c(t) + b(t)[\rho + \delta(1 - q(t))], \quad \forall t \leq T \\
\frac{\dot{c}(t)}{c(t)} &= \frac{\lambda}{q(t)} \left[c(t)W_b^j(b(t)) - 1\right], \quad \forall t \leq T \\
q(T) &= \frac{\lambda\phi}{\rho + \lambda}, \\
c(T) &= y_L + \rho\phi b(T), \\
\log(c(T)) - \log(y_L) &= \lambda \left[W^j(\phi b(T)) - W^j(b(T))\right] - W_b^d(b(T))\dot{b}(T), \\
& b(0) \text{ given,}
\end{aligned}
$$

where $W^j(\cdot)$ and $W^d(\cdot)$ are defined respectively in equation (28) and (29).

The competitive equilibrium is then obtained as a solution of a well-known problem in physics and engineering, called *boundary value problem.* Intuitively, given the solution for the terminal conditions at T, the solution of the system finds a path for the $\dot{b}(t)$, $c(t)$, $q(t)$, and therefore for $b(t)$, that links the terminal conditions to the given initial value $b(0)$ through the equilibrium path.[8]

In the equilibrium path, if the recession is long lasting and income does not jump to the high state before default, the government must issue bonds in order to keep a roughly steady level of consumption. While debt increases, default incentive rises and investors continuously devalue the bond. In turns, a lower bond price requires a larger amount of debt to finance consumption. This spiral continues until the bond price reaches the level $q(T) = \frac{\phi\lambda}{\rho+\lambda}$, at which point the government defaults.

5 Lending Mechanism and Ex-Post Equilibrium

In this section we show that upon default, a balanced budget policy intervention, paid by investors, can improve the market outcome. We first focus on the equilibrium ex-post and show that, at default, existing creditors would have incentive to extend credit to the country at a better price than the market in order to delay the time of default. We propose a simple and tractable policy that incentivizes new investors to do so and we will show how the intervention may significantly affect bond prices and default thresholds.

Policy The policy we consider takes a form of a subsidy on lending financed by taxes on investors. The policy intervention starts at a generic time T^P and eventually ends at time T^E.[9] We postulate that, conditional on the current level of debt b, a policy maker sets a subsidy $g(b)$ per unit of bonds underwritten on the primary market. Since the subsidy is internalized by competitive investors, in equilibrium the borrowing government will

[8] A numerical solution for the boundary value problem can be computed in matlab using the function bvp4c.m. As standard for non-linear system, it is not trivial to prove the existence and the uniqueness of the solution. Nevertheless, for any calibration of the model we have tried, we were able to always find a unique numerical solution.

[9] For convenience in this section we refer to b as the state variable; therefore, it is equivalent to state that there is an asset level $b(T^P)$ at which the policy starts, and an asset level $b(T^E)$ at which the policy ends.

be offered a price equal to:

$$q^p(b) = q(b) + g(b), \quad \forall b \in \left[b(T^P), b(T^E)\right], \tag{13}$$

where $q^p(b)$ is the resulting bond price on the primary market and $q(b)$ is the bond price on the secondary market, which in equilibrium will depend both on the level of the subsidy and on the type of tax used to finance the policy, as it will be clear in the next section. Notice that, for convenience, our notation now implies the current level of asset holding (debt), b, as a state variable. From now on we express the problem in recursive form.

Remark. It is important to understand, as it will be clarified later, that by changing $g(b)$, the policymaker is able to implement any $q^P(b)$ she likes. Therefore, with a slight abuse of notation, in order to simplify the exposition, we will sometime refer to $q^P(b)$ as a policy instrument.

Let $G(b)$ be the gross subsidy, which is also the total cost of the policy, at any debt level during the intervention, i.e.:

$$G(b) = -g(b)(\dot{b} + \delta b), \quad \forall b \in \left[b(T^P), b(T^E)\right], \tag{14}$$

which is the product of the per-unit subsidy and the quantity of new bond issuance. We assume that the intervention, in each period in which it is in place, is balanced budget and financed by levying taxes on all investors, which generate a total amount of tax revenue, $R(b)$.[10] The balance budget condition implies:

$$G(b) = R(b), \quad \forall b \in \left[b(T^P), b(T^E)\right].$$

We will specify how tax revenue is generated in the next section.

How does the policymaker set the subsidy? We postulate a subsidy $g(b) \geq 0$ that makes the borrowing government indifferent to default or keep borrowing. This means that, by construction, the policy makes the borrowing government ex post as well off.

[10] Since it does not effect the main results of this section, we will specify the fiscal policy that generates the tax revenue in the next section.

We then show in the next section that, once accompanied with an optimal stopping time, this policy actually makes creditors always better off ex-post and is, therefore, Pareto improving. Given equation (13) and the fact the that market price $q(b)$ is known, we can equivalently formulate the problem in terms of a policy-maker that sets directly $q^P(b)$ at each point in time. Rewriting the problem in a recursive formulation, the policy can be determined by the solution of the following problem:

$$\text{set} \quad q^p(b): \; W(b|q^p(b)) = W^d(b) \tag{15}$$

$$\text{with} \quad (\rho+\lambda)W(b|q^p(b)) = \max_{c(b)} \; \log(c(b)) + \lambda W^j(b) + W_b(b)\dot{b}, \tag{16}$$

$$s.t. \quad q^p(b)\dot{b}(b) = y_L - c(b) + b[\rho + \delta(1-q^p(b))]. \tag{17}$$

Hence, the policy maker sets the policy $q^p(b)$ that makes the SOE economy indifferent between defaulting and maximing its utility, subject to its budget constraint, while staying in the market and facing the new bond price $q^p(b)$.

Substituting (15) in (16) and taking first order conditions with respect to $c(b)$, the solution of this problem takes the form of a system of three equations in three unknowns $\left\{c(b), \dot{b}(b), q^P(b)\right\}$, i.e.:

$$\log(c(b)) - \log(y_L) = \lambda(W^j(\phi b) - W^j(b)) - W_b^d(b)\dot{b}(b), \tag{18}$$

$$\frac{q^P(b)}{c(b)} = W_b^d(b), \tag{19}$$

$$q^P(b)\dot{b}(b) = y_L - c(b) + b[\rho + \delta(1-q^P(b))]. \tag{20}$$

Given b, the system pins down the policy functions $c(b), \dot{b}(b), q^P(b)$. The first two equations determine the government indifference condition between borrowing and defaulting, while the last equation is the standard government's budget constraint.

The investors' gain and the length of intervention The solution of the dynamic system presented above ignores investors' incentives. Is this type of intervention beneficial for investors? And if so, for how long? On the one hand, bond holders might gain from the government delaying default, but, on the other hand, they have to finance the policy by paying taxes. In this section we quantify the net gain of investors from the policy and, consequently, we pin down the duration of the policy intervention.

For illustrative purposes, in the rest of this section we assume that the policymaker implements the policy at the time in which the government would have defaulted, that is $T^P = T$, and chooses an optimal stopping time T^E for the policy in order to maximize the lifetime utility of a representative investor who holds the entire stock of debt until maturity, finances the policy intervention and underwrites every new bond issuance. Hence, we implicitly assume that the agents do not know about the possibility of policy intervention until the default time T arrives and also that the policy is designed so that the it effectively starts at time T.[11] In the next section we will relax these assumptions. The problem of the policy maker is then given by:

$$V(-b(T)) = \max_{T^E} \int_T^{T^E} [y_L - c(t) - \lambda b(t)] e^{-(\lambda+\rho)(T^E - T)} dt + \frac{\lambda\phi}{\rho+\lambda} b(T^E) e^{-(\lambda+\rho)(T^E-T)}. \quad (21)$$

see Appendix A.8 for the full derivation. Maximizing with respect to T^E, yields the transversality condition of the optimal stopping time problem:

$$y_L - c(b(T^E)) - \lambda b(T^E)(1-\phi) - \frac{\phi\lambda}{\rho+\lambda} \dot{b}(T^E) = 0. \quad (22)$$

this, applied to the system (18)-(20), determines the new debt at default $b(T^E)$ and, given the equation of motion for $\dot{b}$, the corresponding optimal stopping time T^E. Notice that $b(T^E)$ is independent of the time of intervention. The transversality condition states that at the margin, the value of delaying default of one instant should be zero. The value of delaying default, in turn, is the sum of three terms: the net outlays of resources, $y_L - c$, which relates to the cost of the intervention; the option value of being repaid in full if the jump happens at that instant, $-\lambda b(1-\phi)$; and the expected return on the additional assets, $-\frac{\phi\lambda}{\rho+\lambda}\dot{b}$. This suggests that a sufficient condition for an intervention to be warranted at any point in time is that the marginal value for investors of delaying default is positive, the is:

$$IMI(t) \equiv y_L - c(t) - \lambda b(t)(1-\phi) - \frac{\phi\lambda}{\rho+\lambda}\dot{b}(t) \geq 0 \quad (23)$$

[11] As it will be clear in the next section, this is a particular case, since the knowing that the policy will take place might create a market response so that the policy will in fact start at a later time than T. However, it is possible to design the policy for which the market response is nihil. This corner case makes the exposition of this section more intuitive and it will be generalized in the rest of the paper.

where we have denoted with the function $IMI(b)$, the*investors' marginal incentive.*

The following results characterizes the properties of the intervention:

Proposition 4. *The Ex-post intervention.*

1. *If b(T)<0, then the Competitive equilibrium is suboptimal. At T, there always exists an intervention which keeps the SOE indifferent and makes investors strictly better off.*

2. *The length of intervention is limited. Specifically, there exist $T^E < \infty$, and associated debt $b(T^E)$, at which the authority stops the intervention and the SOE government defaults.*

See Appendix A.10 for the proof.

Proposition 4 states two important results. The first one is that if the policy was triggered at time T, then delaying default would make investors better off than if the government was left to default. Hence, the intervention is Pareto improving.[12] The second one relates to the length of the intervention and answer the question: for how long will the intervention continue? The length of intervention depends on how the cost for taxpayers grows with respect to the benefit. The statement (2) implies that the cost increases faster so that the intervention is always bounded in time Recall that the cost of the intervention, $G(\cdot)$, is financed by investors, and it is proportional to the distance between the policy price $q^P(\cdot)$ and the market price $q(\cdot)$, since that distance is also the expected loss on each unit of new bond financing. Now, if $q^P(\cdot)$ is low enough, then the cost of the intervention is relatively small and, therefore, investors' intervention gain that comes from delaying default exceeds its fiscal cost. On the contrary, if the policy price $q^P(\cdot)$ is too high, the fiscal cost of default might exceed the benefit and the authority needs to stop the intervention since it is not anymore in the interest of creditors. But recall also that by Proposition 9 the policy price is always increasing, which means that to keep the SOE as well off, the authority needs to offer continuously better condition. Soon enough the fiscal burden for investors become sufficiently large that the intervention is not anymore beneficial for them, the policymaker stops the policy, and the government defaults.

[12]Recall that by construction the policy leaves the SOE indifferent. Hence, since investors are better off, then the policy is Pareto improving.

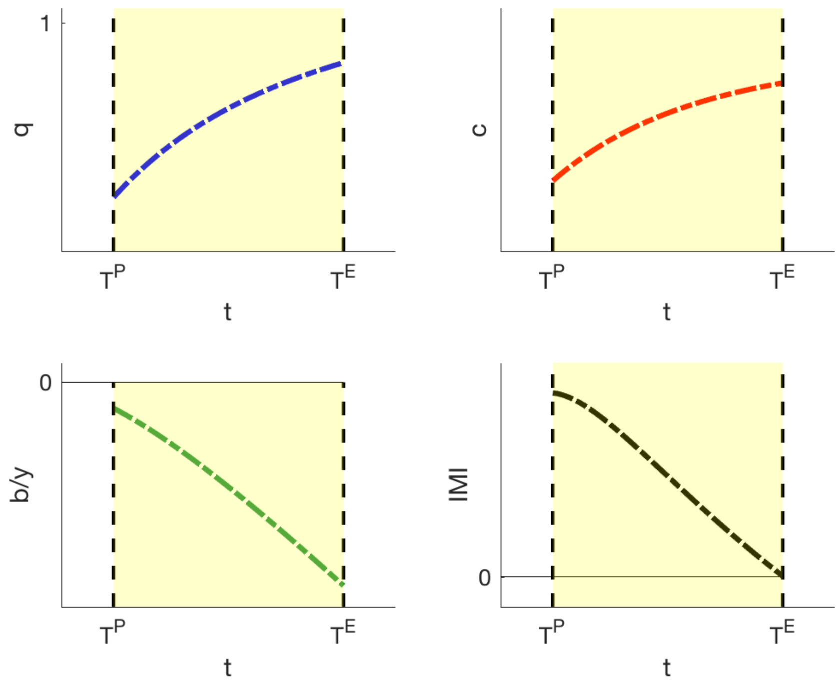
This mechanism reveals an interesting balance of power. If the SOE government has large incentives to default, then the authority is forced to offer a high bond price, which is very costly for taxpayer and the intervention will be very short. On the other hand, if the SOE government has small incentives to default, the intervention is relatively cheap and the investors' are happy to finance the government for a longer time waiting for the good output outcome to realize.

Figure 2 plots the dynamics of the bond price $q(t)$, government's assets $\frac{b(t)}{y_L}$ and consumption $c(t)$, and of the marginal value of delay default (LHS of the transversality condition (22)), assuming that the intervention is called for at time $T^P = T$. When the time of intervention T^P arrives, the policymaker offers an upper sloping price for the government's bond. Better borrowing conditions for the government are welcomed by long-term investors that would have otherwise lost part of their investment due to the upcoming default. At this conditions the government is happy to stay in the market, continuing to borrow and increasing its consumption, while investors are better off since they will continue to receive the interest repayments and default is at least delayed. If the recession is long-lasting, so that the country does not jump to a better state of the economy, the policy continues but become more and more costly for investors. At the time T^E the benefit of the intervention is exactly counterbalanced by that cost, there is no anymore marginal gain for investors to continue to finance the policy, and, therefore, the policymaker stops the intervention, and the country defaults.

The setting provided in this section relies on two strong assumptions: *i*) the intervention is not anticipated either by the government or by the creditors, *ii*) the market does not react to the policy and the intervention occurs at the time the government would have defaulted absent the policy, that is $T^P = T$. In the next section we relax these assumptions.

Discussion on policy implementation The policy design that we proposed is instrumental to the purpose of highlighting the pecuniary externality at the heart of the problem. Our policy is indeed a transfer from existing creditors to new investors that underwrite bonds on the primary market in the current period. It is this transfer that, by decreasing the interest rates for the borrowing government, gives incentives to the

Figure 2 – Bond Price and Debt before Default



Note: this graph plots the competitive equilibrium path of the bond price (top panel), level of asset as a fraction of output (central panel), and SOE's consumption (bottom panel), as a function of time (x-axis). The shaded are denotes the period in which the policy is in place.

government to borrow further. A subsidy on investment is a policy that is not controversial as it can be easily implemented using market instruments. More controversial is be the ability of the policymaker to tax bondholding, in particular when bondholding are dispersed and held internationally. This casts some doubts on the feasibility of the intervention, in particular because absent the possibility to tax bondholders in a distortive manner, if anticipated, the policy can be ex-ante detrimental, as we show in the next section.

Once the mechanism is clear, it is then easy to think to alternative arrangements that can lead to the same outcome. Similarly to our policy, we can think that a credit line is provided directly by an international institution that finances the intervention through taxes. Alternatively, we could think to more nuanced contractual arrangements. For example, we can think to a complex seniority structure that, after some debt level, gives progressively increasing seniority to new bondholders, or equivalently that debt is progressively restructured conditional on the country continuing to borrow from the market.

6 Ex-Ante Analysis

The previous section was useful to show that in our framework there is scope for policy intervention and to explain what are the characteristics of the policy when it is in place. The government may default because the competitive bond price is too low, as it does not reflect the value of delaying default on the existing stock of debt, and investors cannot coordinate to provide a better price.

We proved that, in this case, a policymaker that internalizes the interests of existing creditors has always incentive to intervene ex-post and to extend credit to the government. We also proved that, however, the policy is limited in time, since investors' gain from providing additional financing declines to zero. Nevertheless, it is natural to assume that ex-ante the market reacts to the knowledge of the existence of the policy, and the bond price will incorporate the benefits of future policy intervention. As before, we denote with $b(T^P)$ the debt level at which the policy starts and with $b(T^E)$ the debt level at which the policy ends. However, not necessarily we have that $T^P = T$, as assumed in the ex-post analysis, but the moment in which the policy starts, T^P, will be endogenously determined. We will show that: (i) ex-ante the bond price is indeed affected by the knowledge of the policy; (ii) as a consequence, the endogenous debt level at which the policy intervention is triggered varies with respect to the debt level at which the country would have defaulted absent policy; and (iii) the ex-ante properties of the bond price, and consequently of debt, are crucially a function of the fiscal policy chosen to finance the intervention.

Ex-ante Market Bond Price. First, we investigate how the value of a bond changes, depending on the tax rule in place, when the policy intervention is common knowledge. We assume that the subsidy is financed by taxing the entire population of investors, and that the tax is composed by two components: a proportional tax per-unit of asset and a lump-sum tax, which taxes each investor i independently of asset holding. Restricting attention to symmetric policies, we can then express the aggregate tax revenue, $R(b, \alpha)$, as:

$$R(b,\alpha) \equiv \int_i a^i \tilde{\tau}(b,\alpha)di + \int_i \tau(b,\alpha)di, \quad \forall b \in \left[b(T^P), b(T^E)\right], \tag{24}$$

where $\tilde{\tau}(b,\alpha)$ is the proportional tax per unit of asset, $\tau(b,\alpha)$ is the lump-sum tax per agent-investor i and, α is the fraction of the total tax revenue financed through the proportional tax. We refer to α as the tax rule or fiscal policy. In fact, since the intervention is balanced budget, it must be $G(b,\alpha) = R(b,\alpha)$; therefore, for a given tax rule α, we necessarily have that: $\tilde{\tau}(b,\alpha) = -\frac{\alpha G(b,\alpha)}{b}$ and $\tau(b,\alpha) = (1-\alpha)G(b,\alpha)$. The dependence of the taxes on b captures the fact that the total amount of tax revenue required to finance the intervention varies with the debt level, since the total revenue needs to equate the gross subsidy, as described in equation (14).

Remark. Notice that α could be greater than 1; in this case, the tax rule implies a strong tax on asset holding and, at the same time, a lump-sum transfer to investors. This case will be relevant in the next section.

As an extension to equation (7), we can compute the dynamic equation for the bond price when an intervention policy associated with the tax rule α is expected. The equilibrium bond price in the secondary market, which we denote as $q(b,\alpha)$, reads:

$$(\rho+\delta)q(b,\alpha) = -\tilde{\tau}(b,\alpha) + \rho + \delta + \lambda(1 - q(b,\alpha)) + \dot{q}(b,\alpha), \tag{25}$$

The derivation is shown in the Appendix A.11. Notice that the value of a bond depends on the extent to which taxation is affected by individual portfolio decisions, that is it depends only on the distortive tax component $\tilde{\tau}(b,\alpha)$ and not on the lump-sum tax component $\tau(b,\alpha)$. In fact, if it were, an investor would have an arbitrage opportunity: he could make a gain simply by increasing the amount of bonds in its portfolio and selling short an asset with the same payoff structure of the government bond. Finally, because of its dynamic nature, equation (25) implies that the bond price is affected by the tax rule at any debt level (or equivalently at any time), even before the time T^P, in which the policy actually is implemented.

Ex-Ante Markov Equilibrium. We are now ready to define the ex-ante symmetric equilibrium of our economy when the intervention is anticipated and prior to the jump to the high income state.

Definition 3. Given an asset level, b, a Markov Symmetric Rational Expectation Equilibrium is:

- a tax rule, α;
- a per-unit subsidy policy, $g(b, \alpha)$, a gross subsidy policy $G(b, \alpha)$, and a tax revenue policy $R(b, \alpha)$;
- a bond market price, $q(b, \alpha)$, and a policy bond price $q^P(b)$;
- a consumption policy $c(b)$, a saving policy $\dot{b}(b)$, and an investors' asset holding policy $a(b)$;
- a saving level at which the government triggers the policy intervention $b(T^P)$, and a saving level at which the policy stops the intervention $b(T^E)$;
- a government default value $W^d(b)$, a goverment jump value $W^j(b)$, a government continuation value $W(b|q(b, \alpha))$, and a a government continuation value under the policy $W(b|q^p(b))$

such that:

(i) taking as given the bond market price, $\forall b \leq b(T^P)$, the government continuation value, $W(b|q(b, \alpha))$, solves:

$$\begin{aligned}(\rho + \lambda)W(b|q(b, \alpha)) = \max_{c(b), b(T^P)} & \log(c(b)) + \lambda W^j(b) + W_b(b|q(b, \alpha))\dot{b}(b) \\ s.t. \quad & q(b, \alpha)\dot{b}(b) = y_L - c(b) + b[\rho + \delta(1 - q(b, \alpha))], \\ & W\left(b(T^P)|q(b(T^P), \alpha)\right) = W^d(b(T^P));\end{aligned}$$

(ii) taking as given the bond policy price, $\forall b \in [b(T^P), b(T^E)]$, the government continuation value under the policy $W(b|q^p(b,\alpha))$ solves:

$$(\rho+\lambda)W(b|q^p(b)) = \max_{c(b)} \log(c(b)) + \lambda W^j(b) + W_b(b|q^p(b,\alpha))\dot{b}(b)$$
$$s.t. \quad q^p(b)\dot{b}(b) = y_L - c(b) + b[\rho + \delta(1-q^p(b))];$$

(iii) the default value and the jump value are defined as in equations (29) and (28), respectively;

(iv) the bond price in the secondary market $q(b,\alpha)$ is consistent with investors breaking-even in expectation;

(v) bond markets clear, i.e. $a(b) = -b$;

(vi) the monetary/fiscal authority:

- starts the intervention at $b(T^P)$ and solves its problem in (18)-(20);
- the per unit subsidy $g(b,\alpha)$ is given by $g(b,\alpha) = q^p(b) - q(b,\alpha), \quad \forall b \in \big[b(T^P), b(T^E)\big]$;
- pays a gross subsidy $G(b,\alpha) = -g(b,\alpha)(\dot{b}+\delta b), \quad \forall b \in \big[b(T^P), b(T^E)\big]$;
- follows the policy rule in (24);
- balances its budget, i.e. $R(b,\alpha) = G(b,\alpha)$, $\forall b \in [b(T^P), b(T^E)]$;
- stops the intervention at $b(T^E)$.

With the policy in place, rather than choosing the optimal time of default T, the government chooses the optimal time at which it requires a policy intervention T^P. This, in equilibrium, gives rise to an endogenous level of debt $b(T^P)$ at which the policy is triggered. The policy intervention is characterized by the following proposition:

Proposition 5. Characterization of Policy Intervention. In the ex-ante symmetric rational expectation equilibrium described above, at the time of intervention T^P, it must be:

$$q\left(b(T^P),\alpha\right) = q^P(b(T^P)).$$

Corollary 6. *$b(T^P)$ is increasing in α; $q(b(T^P), \alpha)$ is decreasing in α.*

See Appendix A.12 for the proofs.

This proposition tells us that, at intervention, the price offered by the policy maker should be equal to the market price. The intuition is straightforward: if the market price was higher than the policy price, the government would be better off by borrowing from the market; on the other hand, if the market price was lower than the policy price, then, by continuity of the price function, the government could enjoy better borrowing conditions by requiring the intervention forward in time. This results allow us to define the terminal condition for the government problem and uniquely pin down $b(T^P)$. Proposition 5 also emphasises that the fiscal policy α affects the level of debt (or time) at which the policy starts, $b(T^P)$, but not the level of debt at which it ends $b(T^E)$. The reason is that different fiscal policies affect the path of the bond price in the secondary market, as displayed in equation (25), but not the path of the policy bond price, as clearly stated in the problem (18)-(20). [13]

Moreover, the fact that, given b, the bond price is decreasing in α, implies that the higher the α, the lower the level of debt at which the intervention will be triggered.

7 Welfare

As equation (25) displays, the ex-ante effects on the equilibrium bond price in the secondary market depend on how the individual taxation is linked to the amount of individual asset holdings. Hence, the ex-ante welfare implications of the policy are tightly related to the fiscal rule, α. In this section, we analyze in detail this property and we spell out the condition for the policy to be ex-ante Pareto improving.

In order to properly address the welfare effect of the policy, we perform a counterfactual analysis and we compare a world in which it is known that the policy exists against a world absent any policy intervention. To do that, we define the ex-ante welfare gain of a representative investor that holds all the initial stock of debt $b(0)$ and underwrites every new bond issuance as $\Delta V(\alpha, b(0))$. Similarly, we define the ex-ante welfare gain of the borrowing government as $\Delta W(\alpha, b(0))$. The notation makes explicit that the two

[13] This is the reason why the policy price function $q^p(b)$ does not explicitly depends on α, since the fiscal policy only affect the level of debts at which the policy takes places.

measures of welfare are a function of the relevant fiscal rule α and are defined conditional on a given initial stock of debt $b(0)$. The first step of our analysis is to characterize the link between the fiscal rule α and these welfare measures. The gain from the policy for the representative investor is the sum of two components:

$$\Delta V(\alpha, b(0)) = -b(0) \int_{T}^{T^E} (\rho + \lambda(1-\phi))e^{-\delta T} e^{-(\rho+\lambda)t} dt - (1-\alpha) \int_{T^P}^{T^E} G(b(t), \alpha) e^{-(\rho+\lambda)T^E} dt. \tag{26}$$

The first term is the gain from the delay in default brought about by the fact that the policy will delay default, from time T, when the policy does not exist, to T^E, when the policy exists. To the extent that $T^E > T$, this term is positive. We can interpret this term as a valuation effect; since investors at time 0 are endowed with a given initial stock of assets $a(0) = -b(0)$, their gain from the existence of the policy stems from the fact that these assets are generally worth more when default is less likely. The second term reflects the degree by which the market bond price does not internalize the overall cost of the intervention. In fact, for any $t \in [T^P, T^E]$, the intervention has a total cost for investors equal to $G(b(t), \alpha)$; however, only the fraction of this cost financed by the distortionary tax, α is internalized, while the fraction $(1-\alpha)$ financed by the lump-sum tax is not internalized. If $\alpha < 1$, the bond price does not internalize the fact that investors will have to *pay* a lump-sum tax, while for $\alpha > 1$, the fact that investors will *receive* a lump-sum transfer. In the latter case, the existence of the policy limits the appreciation of the bond price in the market and investors will largely benefit from the policy implementation. That is why the second term in equation (26) is positive only when $\alpha > 1$.

We now restrict the attention to the case in which $a(0) = 0$, which means that we eliminate from the welfare analysis any effect due to pure valuation changes. This allows us to isolate the effects of the fiscal policy on welfare that stem purely from how the market bond price internalizes the policy cost.

Proposition 7. Let $\Delta W(\alpha, 0)$ the welfare gain implied by the existence of the policy for the borrowing country and $\Delta V(\alpha, 0)$ the welfare gain for the representative investor, then:

- *if* $\alpha \leq 1$*, then* $\Delta V(\alpha, 0) \leq 0$*, with* $\Delta V(1, 0) = 0 \iff \alpha = 1$*;*

- $\forall \alpha \geq 1$*,* $\Delta V(\alpha, 0)$ *is monotonically increasing in* α*.*

- $\forall \alpha \geq 1$*,* $\Delta W(\alpha, 0)$ *is monotonically decreasing in* α

See Appendix A.13 for the proof.

Proposition 7 states a very important result. The monetary/fiscal authority can use the fiscal policy α as redistribution instrument between investors' welfare and the SOE's welfare. Financing the intervention by imposing heavy *proportional* taxes to bond holding depresses bond prices and the country's welfare in favor of investors' welfare (note that for $\alpha \geq 1$, despite paying high proportional taxes, investors receive a lump sum transfer from the monetary/fiscal authority). On the contrary, by attenuating the dependence of the tax upon bond holding, the policymaker creates an over valuation of the bond price which diminishes investors' welfare in favor of the country's welfare by creating an implicit fiscal transfer to the latter. Indeed, as long as the policy has always been in place (which corresponds in our case to an announcement at $b(0) = 0$, for $\alpha < 1$ a policy is never Pareto improving for investors.[14]

This proposition indeed shows that α is actually a measure of the implicit transfer from investors to the country.

The second step is to characterize the set of Pareto improving interventions.

At the limiting case in which the fiscal rule aggressively taxes bond holding so that $q(T^P) = q(T)$, the welfare gain of the SOE must be equal to zero. In this case the existence of the policy does not affect the market price before intervention and therefore, its effects are equivalent to the effects of an ex-post policy that takes place at T, exactly as studied in Section 5. On the other hand, investors will get the entire benefit from the intervention as in the case of an intervention ex-post. On the opposite extreme, in which $\alpha = 1$, by equation (26), it must be $\Delta V(\alpha, 0) = 0$, and all the benefit from the intervention will go to the SOE.

In addition, when α decreases, then, by Proposition 7 investors' gain from intervention decreases and the country's gain increases. That means that all the policy

[14]Of course, if the policy was not known at the onset and is announced at $b(0) < 0$, the gain for existing investors is higher, as equation (26) makes clear, which justifies an intervention with $\alpha < 1$. We abstract from this case.

characterized by α such that the investor's gain is still positive are Pareto improving. The following Proposition formalizes this result.

Proposition 8. ***Pareto set.*** *For* $b(0) = 0$*, there exists a non empty set* $\alpha \in [1, \bar{\alpha}]$ *of Pareto Improving policies. In particular,* $\Delta W(1, 0) > 0$ *and* $\Delta V(1, 0) = 0$*, while* $\Delta W(\bar{\alpha}, 0) = 0$ *and* $\Delta V(\bar{\alpha}, 0) > 0$*.*

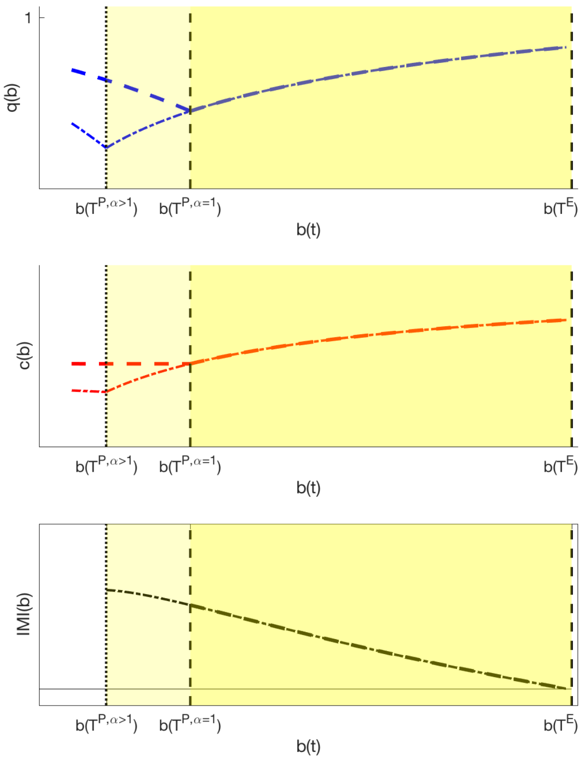
See Appendix A.14 for the proof.

In Figure 3 we display the equilibrium paths of the policy that are at the boundary of the Pareto improving set. We show the two extreme cases. The first case is characterized by a fiscal policy that taxes aggressively bond holding; in this scenario $\alpha = \bar{\alpha} > 1$, and it is displayed by thin dash-dotted lines. As stated in Proposition 8, this scenario is characterized a positive gain for investors and by the smallest, and equal to zero, gain for the SOE. The time of intervention in this case is indicated with $T^{P,\alpha>1}$ and the intervention region includes the light and dark shaded areas. The second case is characterized by a fiscal policy that taxes less aggressively bond holdings. In this scenario $\alpha = 1$, and it is displayed by thick dashed lines. This scenario is characterized by a large gain from the policy for the SOE and by a gain for investors equal to zero. The time of intervention in this case is indicated with $T^{P,\alpha=1}$ and the intervention region is represented with a dark shaded area. Any fiscal policy such that $\alpha \in [1, \bar{\alpha}]$ belong to the intervals generated by the two cases displayed in the figure and they represent a Pareto improving policy.

The least aggressive bond-holding tax, within the Pareto set, $\alpha = 1$, increases bonds' evaluation and this delays the time of intervention, to $T^{P,\alpha=1}$. Better market prices increase the country's consumption, and diminishes the ex-post incentive to intervene for investors, as indicated by the IMI panel. On the contrary, the most aggressive bond-holding tax, within the Pareto set, $\alpha = \bar{\alpha}$, does not generate any price appreciation before the intervention. This is the case in which the policy starts at the same period in which the SOE would have had defaulted, if the policy were not introduced at all, that is $T^{P,\bar{\alpha}} = T$.[15] Also, the lower is the level of α, the largest is the size of that bond price appreciation, and the largest is the redistribution of welfare from investors to the SOE.

[15]For clarity, this is the corner case we have considered in our ex-post analysis conducted in Section 5.

Figure 3 – Distortive taxation and Pareto Improving Policies



Note: this graph plots the competitive equilibrium path of the bond price (top panel), SOE's consumption (central panel), and the IMI values (bottom panel) as a function of bond holding. The thin dash-dotted lines represent the equilibrium under the fiscal policy $\alpha = \bar{\alpha} > 1$. The thick dashed lines represent the equilibrium under the fiscal policy $\alpha = 1$.

8 Conclusions

This paper highlights the presence of a novel externality in sovereign bond markets, that we label as *non-exclusivity externality*. This externality arises because in a competitive market, in which the ownership of debt is anonymous and dispersed, the equilibrium price of new debt on the primary market might be too low to avoid default, even though preventing default would be in the interest of existing creditors. We then show that a policy that subsidizes the underwriting of new bond issuance by taxing existing bondholders, is ex-post Pareto improving. This is true because, upon default, the benefit from delaying default is always greater than the cost, even though the duration of the policy is limited in time. However, the benefit ex-ante depends crucially on the ability of the policy-maker to tax bond-holdings in such a way that the price ex-ante will reflect the cost of the policy. Absent this possibility, the policy is ex-ante detrimental.

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A Appendix

A.1 Proof of Proposition 1

Proof. In the competitive equilibrium we have that $q = p$ and the period-1 default condition is: $W_1^D \geq W_1^{ND}(B_1, p)$. The optimal asset decision, evaluated at $q = p$ is: $B_2^*(B_1, p) = \frac{(y_L+B_1)-y_H}{1+p}$. Substituting into the non-default value in period 1 and doing some simple algebra, the government decides to default if and only if:

$$\frac{\log(y_L) + p\log(y_H)}{1+p} \geq \log\left(\frac{y_L + py_H + B_1}{1+p}\right). \tag{27}$$

Assume B_1=0, by Jensen's inequality, the concavity of the logarithm function implies that equation (27) is not satisfied and therefore a necessary condition for the government to default is that $B_1 < 0$. Since the RHS of (27) is monotonically increasing and continuous in B_1, there exists a unique threshold

$$\bar{B}_1 = (1+p)\left(y_L y_H^p\right)^{\frac{1}{1+p}} - (y_L + py_H) < 0$$

such that the government defaults if and only if $B_1 \leq \bar{B}_1$. □

A.2 Proof of Proposition 2

Proof. The equilibrium price of the bond under the subsidy is: $\bar{q} = p + \xi$. Investors internalize the subsidy and break even in expectation. The welfare of the country is increasing in ξ since the new price relaxes the government budget constraint. Are existing investors now willing to finance the subsidy? With a strictly positive subsidy ξ, the country will not default. Their welfare gain from introducing the subsidy is:

$$V^{oldI}(p+\xi) - V^{oldI}(p) = -B_1 \mathbb{1}_{\xi>0} - \xi(-\tilde{B}_2)$$

The first term is the revenue in period 1 that occurs only when ξ is strictly positive, since only in that case lenders will get back their original investment; the second term is the cost of the transfer, which is equal to the unit cost of the subsidy, ξ, and the total amount of new bond optimally sold by the country and acquired by new investors, equal to $-\tilde{B}_2$. Taking the right hand side limit as $\xi \to 0^+$ of the expression above, we have that

$$\lim_{\xi \to 0^+} \left[V^{oldI}(p+\xi) - V^{oldI}(p)\right] = -B_1$$

This discontinuity and the fact that the welfare gain is continuous in δ prove the result □

A.3 Definition of a Continuous-time Markov chain

Definition 4. Continuous-time Markov chain. A continuous-time Markov chain with finite or countable state space $\mathcal{Y}$ is a family $\{Y_t = Y(t)\}_{t\geq 0}$ of $\mathcal{Y}$-valued random variables such that:

(a) The paths $t \mapsto Y(t)$ are right-continuous step functions; and

(b) For any set of times $t_i < t_{i+1} = t_i + s_{i+1}$ and states $y_i \in \mathcal{Y}$, with $t_0 = 0$,

$$P\left(Y(t_{k+1})|Y(t_i) = y_i \forall i \leq k\right) = P\left(Y(s_{k+1}) = y_{k+i}|Y(0) = y_k\right).$$

Condition (a) guarantees that the Markov chain makes only finitely many jumps in any finite time interval. Condition (b) is the natural continuous-time analogue of the Markov property. It requires two things: first, that the future is conditionally independent of the past given the present, and second, that the transition probabilities are time-homogeneous.

A.4 Proof of Proposition 3

1. Proof of 1. The Hamiltonian-Jacobi-Bellman equation associated with the investors' problem in (4) is;

$$(r+\lambda)V(a,q) = \max_{\dot{a}} \left[-q(\dot{a}+\delta a) + (r+\delta+\lambda)a + V'_a\dot{a} + V'_q\dot{q}\right],$$

where we have dropped the time indexes for simplicity of notation. Notice that the assumption of perfect competition and the fact that investors are atomistic implies that $V(a,q) = \tilde{V}(q)a$, which means that the unit value of an asset must be independent of the quantity of asset holdings.

Then, we have:

$$(r+\lambda)\tilde{V}(q) = \max_{\dot{a}} \left[-(q-\tilde{V}(q))\frac{\dot{a}}{a} + (r+\delta(1-q)+\lambda) + \tilde{V}_q'\dot{q} \right].$$

If $q > \tilde{V}(q)$, the price of the asset would be larger than its value and the investors would like to sell an arbitrarily large number of assets. Viceversa, if $q < \tilde{V}(q)$, the price of the asset would be lower than its value and the investors would demand an infinite number of assets. It follows that in equilibrium it must be that $q = \tilde{V}(q)$. Substituting this relationship in the above expression we obtain statement 1 of the Proposition.

2. Proof of 2. The value of a bond one instant before default is

$$\begin{aligned} q(T-dt) &= \int_{T-dt}^{T} (r+\delta+\lambda)e^{-(r+\delta+\lambda)(s-T+dt)}ds + \int_{T}^{\infty} \phi\lambda e^{-(r+\lambda)(s-T)}ds \\ &= 1 - e^{-(r+\delta+\lambda)dt} + \frac{\lambda\phi}{r+\lambda} \end{aligned}$$

taking the limit for $dt \to 0$ we get $q(T) = \frac{\lambda\phi}{r+\lambda}$.

3. Proof of 3. Since it is never optimal for the government to default in the high income state, the value of a bond solves

$$q(t) = \int_{t}^{\infty} (r+\delta)e^{-(r+\delta)(s-t)}ds$$

and $q(t) = 1$, $\forall t \geq T^j$.

A.5 Value at the jump and at the default

The value at the jump. Let us first derive the value of the government after uncertainty is resolved, i.e. when income jumps to the absorbing high state. In order to obtain analytical results, we assume that the instantaneous utility is $u(c) = \log(c)$ and that the risk-free rate in the economy is equal to the discount factor, $r = \rho$. These assumptions imply that after the jump, since there is no uncertainty, the government will optimally maintain a constant consumption.

If the government has not defaulted prior to the jump, the problem is:

$$\begin{aligned} W^j\left(b(T^j)\right) &= \max_{\{c(t)\}_{t\geq T^j}} \int_{T^j}^{\infty} e^{-\rho(t-T^j)} \log(c(t))dt \\ \text{s.t. } \dot{b}(t) &= y_H - c(t) + (\rho+\delta)b(t) - \delta b(t), \end{aligned}$$

where we have used the fact that after the jump $q(t) = 1$, $\forall t \geq T^j$. The solution of this trivial problem gives the value at the moment of the jump, that is:

$$W^j\left(b(T^j)\right) = \frac{\log(y_H + \rho b(T^j))}{\rho}.$$

If the government has already defaulted prior to the jump, the problem is identical beside the fact that at the moment of the jump the country reenters the financial market with a level of assets that it is a fraction ϕ of its obligation at the moment of default, $b(T)$. Hence, it will start the period of the jump T^j with a level of assets equal to $\phi b(T)$.

Therefore, defining with x the starting level of assets at the time of jump T^j, we can conveniently write the value of the government at T^j as:

$$W^j(x) = \frac{\log(y_H + \rho x)}{\rho} \quad \text{with:} \begin{cases} x = b(T^j) & \text{if } T^j \leq T, \\ x = \phi b(T) & \text{if } T^j > T. \end{cases} \tag{28}$$

The value at default. If the government defaults at time T, then it will remain in autarky consuming the low level of income until the period of the jump, at which point it enjoys the value $W^j(\phi b(T))$ as measured above. Hence, the value function at default as a function of the level of asset $b(T)$ is:

$$W^d(b(T)) = \frac{\log(y_L) + \lambda W^j(\phi b(T))}{\rho + \lambda} \tag{29}$$

A.6 Derivation of the continuous time Euler Equation in equation (9)

Define the current value Hamiltonian:

$$H(b, p, c, t) = u(c(t)) + \lambda W^j(b(t)) + p(t)\dot{b}(t),$$

where $p(t)$ is the costate variable, $b(t)$ is the state variable, $c(t)$ is the control variable and $u(c)$ is a generic utility function which satisfies Inada conditions. The first order conditions of the optimal control problem are

$$\begin{aligned} H_c &= 0, \\ -H_b &= \dot{p}(t) - (\rho + \lambda)p(t), \\ H_p &= \dot{b}(t). \end{aligned}$$

Substituting the derivatives of the Hamiltonian

$$
\begin{aligned}
q(t)u'(c(t)) &= p(t), \\
\lambda W_b^j(b(t)) + p(t)\left[\frac{\rho+\delta}{q(t)} - \delta\right] &= -\dot{p}(t) + (\rho+\lambda)p(t), \\
\dot{b}(t) &= \frac{1}{q(t)}\left(y_L - c(t) + (\rho+\delta)b(t)\right) - \delta b(t),
\end{aligned}
$$

and consolidating the first two equations:

$$
\begin{aligned}
\lambda W_b^j(b(t)) + (\rho+\delta)u'(c(t)) - \delta q(t)u'(c(t)) = &- \dot{q}(t)u'(c(t)) - q(t)u''(c(t))\dot{c}(t) + \\
&+ (\rho+\lambda)q(t)u'(c(t)). \qquad (30)
\end{aligned}
$$

Substituting for $\dot{q}(t) = (q(t)-1)(\rho+\delta+\lambda)$ and using the fact that with log-utility $-\frac{u''(c(t))c(t)}{u'(c(t))} = 1$ and $u'(c(t)) = \frac{1}{c}$ we obtain (9)

A.7 Derivation of the Terminal conditions in equation (10)-(12)

Problem (5)-(8) requires a simultaneous determination of optimal control and terminal time. These problems are usually called *free terminal time problems* and the necessary optimality condition for the terminal time requires the derivation of an additional transversality condition (see Hartl and Sethi (1983) for the formal derivation). Let T be the terminal time and $S(b(T), T)$ denote the *salvage value function*:

$$
S(b(T), T) \equiv W^d(b(T))e^{-(\lambda+\rho)(T)}.
$$

At the optimum terminal time, T, the costate variable must satisfy:

$$
p(T) = S_b(b(T), T),
$$

while the transversality condition is given by

$$
H(b(T), p(T), c(T), T) + S_T(b(T), T) = 0.
$$

The transversality condition requires that at the optimal terminal time, the benefit of delaying default of one instant, given by the Hamiltonian evaluated at T, is equal to opportunity cost of delaying default, given by the derivative of the salvage function with respect to T. Together with the budget constraint, the terminal conditions of the problem define a system of three equations:

$$\log(c(T)) + \lambda W^j(b(T)) + p(T)\dot{b}(T) = (\rho+\lambda)W^d(b(T))$$
$$p(T) = W^d_b(b(T)),$$
$$\dot{b}(T) = \frac{1}{q(T)}\left[y_L - c(T) + (\rho+\delta)b(T)\right] - \delta b(T).$$

Using the fact that $p(T) = \frac{q(T)}{c(T)}$ and using (29), we get

$$\log(c(T)) - \log(y_L) = \lambda\left[W^j(\phi b(T)) - W^j(b(T))\right] - W^d_b(b(T)\dot{b}(T)$$
$$\frac{q(T)}{c(T)} = W^d_b(b(T))$$
$$\dot{b}(T) = \frac{1}{q(T)}\left[y_L - c(T) + (\rho+\delta)b(T)\right] - \delta b(T).$$

By substituting $W^d_b(b(T))$ from equation (29), $W^j_b(b(T))$ from equation (28) and $q(T)$ in (8), the system of equations simplifies to

$$\log(c(T)) - \log(y_L) = \lambda\left[W^j(\phi b(T)) - W^j(b(T))\right] - W^d_b(b(T))\dot{b}(T)$$
$$c(T) = y_H + r\phi b(T),$$
$$\dot{b}(T) = \frac{\rho+\lambda}{\lambda\phi}\left[y_L - c(T) + (\rho+\delta)b(T)\right] - \delta b(T).$$

A.8 Derivation of investors' ICC

The lifetime utility of a representative investor who holds the entire stock of debt until maturity, underwrites every new bond issuance and finances the policy intervention is

$$V(-b(t)) = \int_T^{T^E}\left[-G(b(t)) + q(t)\left(\dot{b}(t) + \delta b(t)\right) - (\rho+\delta+\lambda)b(t)\right]e^{-(r+\lambda)(t-T)} + V(-b(T^E))e^{-(T^E-T)},$$

where we have now incorporated the fact that the cost of the policy, $G(b)$, is a burden for investors. Substituting the expression for $G(b)$ in equation (14), and using the budget constraint of the government post intervention in equation (20), the expression simplifies to (21).

A.9 Auxiliary result: Existence of a steady state for the system (18)-(20)

We now prove that the system (18)-(20) has a unique stable steady state.

Proposition 9. *Let us denote with* $\dot{b}(b)$ *the solution of the saving rate as a function of the level of assets resulting from the system* (18)-(20)*. And assume a solution of the non-linear system above does exist.*

If $b(T) < 0$, then there is a unique stable steady state at which the dynamic system (18)-(20) converges and all the variables remain constant. Moreover, the intervention is characterized by a bond price $q^P(b)$ that is monotonically decreasing in b or, equivalently, monotonically increasing in time.

Proof. Equations (18)-(20), define a system of three equations in three unknowns, $\dot{b}, q^P, c$, given b. At the steady state $\dot{b} = 0$ the system simplifies to:

$$\begin{cases} c = y_L \left(\frac{y_H + \rho\phi b}{y_H + \rho b}\right)^{\frac{\lambda}{r}} \\ q^P = \frac{\phi\lambda}{\rho+\lambda}\left(\frac{c}{y_H + \rho\phi b}\right) \\ c - y_L = b\left[\rho + \delta(1 - q^P)\right] \end{cases}$$

Unfortunately, as standard for a system of non-linear equations, we cannot prove the existence of a steady state and, therefore, we have to rely upon numerical solutions. However, provided that a steady state does exist, we can still study its stability properties. Equations (18)-(20), define an autonomous system of three equations in three unknowns, $\dot{b}, q^P, c$, given b. We use the notation $W_b^j(\cdot)$ and $W_{bb}^j(\cdot)$ to denote respectively the first and the second derivative with respect to b of the function $W^j(\cdot)$. Taking derivatives of each equation in the system (18)-(20) with respect to b we obtain

$$\underbrace{\begin{bmatrix} \frac{1}{c} & \frac{\lambda}{\rho+\lambda}W_b^j(\phi b) & 0 \\ -\frac{\lambda}{\rho+\lambda}W_b^j(\phi b) & 0 & 1 \\ 1 & q^P & \dot{b} + \delta b \end{bmatrix}}_{\equiv A} \begin{bmatrix} \frac{\partial c}{\partial b} \\ \frac{\partial \dot{b}}{\partial b} \\ \frac{\partial q^P}{\partial b} \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda\left(W_b^j(\phi b) - W_b^j(b)\right) - \frac{\lambda}{\rho+\lambda}W_{bb}^j(\phi b)\dot{b} \\ \frac{\lambda}{\rho+\lambda}W_{bb}^j(\phi b)c \\ \rho + \delta(1 - q^P) \end{bmatrix}}_{\equiv v}. \tag{31}$$

The determinant of A is

$$\begin{aligned} det(A) &= -\frac{q^P}{c} + \frac{\lambda}{\rho+\lambda}W_b^j(\phi b)\left(1 + \frac{\lambda}{\rho+\lambda}W_b^j(\phi b)(\dot{b} + \delta b)\right) \\ &= \left(\frac{\lambda}{\rho+\lambda}W_b^j(\phi b)\right)^2(\dot{b} + \delta b). \end{aligned}$$

where the second equality is obtained substituting for equation (19). Let A_2 be the matrix obtained by substituting the second column in A with the vector v, the determinant of A_2 reads

$$\begin{aligned} det(A_2) = &\left(\frac{\lambda}{\rho+\lambda}W_{bb}^j(\phi b)(\dot{b} + \delta b) - (\rho + \delta(1 - q^p))\frac{1}{c}\right) + \\ &+ \left(\lambda\left(W_b^j(\phi b) - W_b^j(b)\right) - \frac{\lambda}{\rho+\lambda}W_{bb}^j(\phi b)\dot{b}\right)\left(1 + \frac{\lambda}{\rho+\lambda}W_b^j(\phi b)(\dot{b} + \delta b)\right) \end{aligned}$$

By Cramer rule, $\frac{\partial \dot{b}}{\partial b} = \frac{det(A_2)}{det(A)}$. A steady state is stable if and only if the derivative $\frac{\partial \dot{b}}{\partial b}$ evaluated at the steady state is negative, formally: $\frac{\partial \dot{b}}{\partial b}|_{\dot{b}=0} < 0$. Notice that at the steady state $\dot{b} = 0$, and, therefore, $det(A) < 0$ since we are restricting our domain of interest on $b < 0$. Stability follows if we can show that at the steady state $det(A_2|\dot{b}=0)>0$.

$$\begin{aligned} det(A_2|\dot{b}=0) = & -\lambda\left(W_b^j(b)-W_b^j(\phi b)\right)-(\rho+\delta(1-q^P))\frac{1}{c}+ \\ & +\frac{\lambda}{\rho+\lambda}W_{bb}^j(\phi b)\delta b+\frac{\lambda^2}{\rho+\lambda}\left(W_b^j(\phi b)-W_b^j(b)\right)W_b^j(\phi b)\delta b. \end{aligned} \tag{32}$$

The envelope condition associated to the government problem which can be derived by taking derivatives with respect to b of the Hamiltonian-Jacobian-Bellman equation in (18), reads

$$\left(W_{bb}^d-\frac{q^{P\prime}(b)}{q^P(b)}\right)\dot{b}=-\frac{W_b^d}{q}\left[\rho+\delta(1-q^P)\right]-\lambda\left(W_b^j(\phi b)-W_b^j(b)\right), \tag{33}$$

and implies that at the steady state

$$\frac{W_b^d}{q}\left[\rho+\delta(1-q^P)\right]-\lambda\left(W_b^j(\phi b)-W_b^j(b)\right)=0$$

Substituting the FOCs of the planner problem, $\frac{q^P(t)}{c(t)}=W_b^d$, it follows that we can simplify $det(A_2|\dot{b}=0)$ to

$$det(A_2|\dot{b}=0)=\frac{\lambda}{\rho+\lambda}W_{bb}^j(\phi b)\delta b+\frac{\lambda^2}{\rho+\lambda}\left(W_b^j(\phi b)-W_b^j(b)\right)W_b^j(\phi b)\delta b>0.$$

The sign follows immediately from the fact that $W_{bb}^j(\phi b)<0$, $\left(W_b^j(\phi b)-W_b^j(b)\right)<0$, $W_b^j(\phi b)>0$ and $b<0$. This proves that if a steady state does exist, it must be stable. In addition, since the inequality $\frac{\partial\dot{b}}{\partial b}|_{\dot{b}=0}>0$ is satisfied for any possible steady state in the domain $b<0$, it must be the case that if a steady state exists it must also be unique on this domain. From (33), we must also have that on the domain of interest where $\dot{b}<0$, $\frac{q^{P\prime}(b)}{q^P(b)}<0$. That because the RHS of (33) is negative for every q^P lower than the steady state level, and in order for the LHS to be negative, since $W_{bb}^d<0$, it must be $\frac{q^{P\prime}(b)}{q^P(b)}<0$. Hence, the policy price is decreasing in b, or equivalently, increasing in t.

□

A.10 Proof of Proposition 4

Define the function

$$ICC(b)\equiv y_L-c(b)-\lambda b(1-\phi)-\frac{\phi\lambda}{\rho+\lambda}\dot{b} \tag{34}$$

this is the LHS of the transversality condition (22), and is the marginal gain for investors from delaying default of one instant. Therefore, $ICC(b)>0$ represents a sufficient condition for an intervention to be Pareto improving, i.e. $ICC(b)\geq 0\Rightarrow V(-\bar{b})>V^d(-\bar{b})$.

1. Proof of 1.

At the time of intervention T, $q^P(b(T)) = \frac{\phi\lambda}{\rho+\lambda}$. Therefore, the government budget constraint in equation (34) reads

$$\frac{\phi\lambda}{\rho+\lambda}\dot{b}(T) = y_L - c(T) + \left(\rho + \delta\left(1 - \frac{\phi\lambda}{\rho+\lambda}\right)\right) b(T).$$

Replace the equation above into the last term of the ICC in equation (34) and evaluate the ICC at T:

$$y_L - c(T) - \lambda(1-\phi)b(T) - \left[y_L - c(T) + \left(\rho + \delta\left(1 - \frac{\phi\lambda}{\rho+\lambda}\right)\right) b(T)\right].$$

Simplifying, it becomes:

$$-\left[\lambda(1-\phi) + \left(\rho + \delta\left(1 - \frac{\phi\lambda}{\rho+\lambda}\right)\right)\right] b(T) > 0.$$

Since all the coefficients are positive, and ϕ and λ are less then one, the term in square bracket is positive. Therefore, $ICC(b(T)) > 0$ at time T whenever the government defaults with some debt $b(T) < 0$. $ICC(b) > 0$ represents a sufficient condition for an intervention to be Pareto improving.

2. Proof of 2.

Denote $\bar{b}$ the steady state level of debt, such that $\dot{b}(\bar{b}) = 0$. Note that i) If $ICC(\bar{b}) \geq 0$, then the intervention will continue indefinitely until the jump to the high income state. ii) If $ICC(\bar{b}) < 0$, by the intermediate value theorem, it must exists $b(T^E) \in [b(T), \bar{b}]$, and associated $T^E < \infty$ such that $ICC(b(T^E)) = 0$. We will show that, if a steady state does exists, then it must be $ICC(\bar{b}) < 0$.

The system of equations (18)-(20) at the steady state reads:

$$\begin{cases} c = y_L \left(\frac{y_H + \rho\phi\bar{b}}{y_H + \rho\bar{b}}\right)^{\frac{\lambda}{\rho}}, \\ q^P = \frac{\phi\lambda}{\rho+\lambda}\left(\frac{c}{y_H + \rho\phi\bar{b}}\right), \\ c - y_L = \bar{b}[\rho + \delta(1 - q^P)]. \end{cases}$$

This proof consists of two parts.

Part 1. First we show that at the steady state $\bar{b}$, it must be $\bar{b} < \frac{y_L - y_H}{\rho}$. Let ϵ be any real constant such that $\bar{b} = \frac{y_L - y_H}{\rho} + \epsilon$, and rearrange that expression as

$$y_H + \rho\bar{b} = y_L + \rho\epsilon$$

or equivalently,

$$y_H + \rho\phi\bar{b} = [\phi y_L + (1-\phi)y_H] + \phi\rho\epsilon.$$

Define the variable ζ as:

$$\zeta \equiv \frac{y_H + \rho\phi\bar{b}}{y_H + \rho\bar{b}} = \frac{[\phi y_L + (1-\phi)y_H] + \phi\rho\epsilon}{y_L + \rho\epsilon}.$$

Notice that $\zeta > 1$ since $\phi \leq 1$ and $\bar{b} < 0$. We can now restate the system in terms of ζ and ϵ as

$$\begin{cases} c = y_L \zeta^{\frac{\lambda}{\rho}}, \\ q^P = \left(\frac{\phi\lambda}{\rho+\lambda}\right) \frac{y_L \zeta^{\frac{\lambda}{\rho}}}{[\phi y_L + (1-\phi)y_H] + \phi\rho\epsilon} \\ y_L \zeta^{\frac{\lambda}{\rho}} - y_L = \left(\frac{y_L - y_H + \epsilon\rho}{\rho}\right)\left(\rho + \delta(1-q^P)\right). \end{cases}$$

By substituting q^P in the third equation:

$$\zeta^{\frac{\lambda}{\rho}} \left[y_L + \left(\frac{y_L - y_H + \epsilon\rho}{\rho}\right) \frac{\delta\phi\lambda y_L}{(\rho+\lambda)\left[(\phi y_L + (1-\phi)y_H) + \phi\rho\epsilon\right]} \right] = \\ = y_L + \left(\frac{y_L - y_H + \epsilon\rho}{\rho}\right)(\rho + \delta).$$

Now, $\zeta > 1$ and $y_L + \rho\epsilon < y_H$, therefore a necessary condition for the equality to be satisfied is that

$$\frac{\delta\phi\lambda y_L}{(\rho+\lambda)\left[(\phi y_L + (1-\phi)y_H) + \phi\rho\epsilon\right]} > \delta + \rho,$$

rearranging the inequality

$$-\frac{\rho\phi(\delta+\rho+\lambda)}{\rho+\lambda} y_L - (\delta+\rho)(1-\phi)y_H > \phi\rho\epsilon.$$

which implies $\epsilon < 0$.

Part 2. We now show that at the steady state, the ICC is not satisfied (i.e. $ICC(\bar{b}) < 0$). Write the first equation of the system in logs as:

$$ln(c) - ln(y_L) = \frac{\lambda}{\rho}\left[ln(y_H + \rho\phi\bar{b}) - ln(y_H + \rho\bar{b})\right].$$

From part 1, since $\zeta > 1$, at the steady state $c \geq y_L$ which implies, from the government budget constraint evaluated at the steady state (third equation of the system), that $q^P \geq \frac{\delta+\rho}{\delta}$. Then

$$c > y_H + \rho\phi\bar{b}.$$

Moreover, from part 1, the fact that $\epsilon < 0$, implies

$$y_L > y_H + \rho\bar{b}.$$

By strict concavity of the logarithmic function (the result is proved in Lemma 10 below), we have:

$$\frac{ln(c) - ln(y_L)}{c - y_L} < \left[\frac{ln(y_H + \rho\phi\bar{b}) - ln(y_H + \rho\bar{b})}{\rho(\phi - 1)\bar{b}}\right].$$

Substituting $ln(c) - ln(y_L) = \frac{\lambda}{\rho}[ln(y_H + \rho\phi\bar{b}) - ln(y_H + \rho\bar{b})]$, we have:

$$c - y_L > \lambda(\phi - 1)\bar{b}.$$

Hence, the ICC is negative at the steady state. Intuitively, the cost of avoiding default, $c - y_L$ is higher than the benefit for the investors $\lambda(\phi - 1)\bar{b}$.

Lemma 10. *Let $C \to R$ be an open interval, $f : C \to R$ is concave iff for any $a, b, c, d \in C$, with $a < b < c < d$,*

$$\frac{f(c) - f(a)}{c - a} \geq \frac{f(d) - f(b)}{d - b}.$$

Proof. We first show that:

$$\frac{f(c) - f(a)}{c - a} \geq \frac{f(d) - f(a)}{d - a}.$$

Suppose that f is concave and take any $a, b, c, d \in C$, $a < b < c < d$. Since $(c - a) > 0$ and $(d - a) > 0$, the expression above holds iff:

$$\frac{f(c) - f(a)}{c - a} \geq \frac{f(d) - f(a)}{d - a},$$

which holds iff (collecting terms in $f(c)$),

$$f(c) \geq \left(1 - \frac{c - a}{d - a}\right) f(a) + \left(\frac{c - a}{d - a}\right) f(d).$$

Since f is concave, the latter holds taking $\theta = \left(\frac{c-a}{d-a}\right) \in (0, 1)$. Moreover, verifying that $c = (1 - \theta)a + (\theta)d$, any function that satisfies the equation needs indeed to be concave.

$$f(\theta d + (1 - \theta)a) \geq \theta f(d) + (1 - \theta)f(a).$$

Similarly we can show that:

$$\frac{f(d)-f(a)}{d-a} \geq \frac{f(d)-f(b)}{d-b}.$$

Collecting terms in $f(b)$,

$$f(b) \geq \left(1-\frac{d-b}{d-a}\right) f(d) + \left(\frac{d-b}{d-a}\right) f(a).$$

The previous proof goes through, taking $\theta = \left(\frac{d-b}{d-a}\right)$, and verifying that, indeed, $b = (1-\theta)d + \theta a$. □

A.11 Derivation of equation (25)

The Hamiltonian-Jacobi-Bellman equation associated with the investors' problem in (4) is;

$$(r+\lambda)V(a,q,b) = \max_{a} \left[-a\tilde{\tau}(b,\alpha)di - \tau(b,\alpha)di - q(\dot{a}+\delta a) + (\rho+\delta+\lambda)a + V_a'\dot{a} + V_q'\dot{q}\right],$$

where we have dropped the time indexes for simplicity of notation. Notice that the assumption of perfect competition and the fact that investors are atomistic requires a solution of the form $V(a,b,q) = \tilde{V}(b,q)a + x(b)$, which means that the unit value of an asset must be independent of the quantity of asset holdings. Then, we have:

$$(r+\lambda)\left[\tilde{V}(b,q) + \frac{x(b,q)}{a}\right] = \max_{a}\left[-(q-\tilde{V}(b,q))\frac{\dot{a}}{a} + (r+\delta(1-q)+\lambda) + \tilde{V}_q'\dot{q} - \tilde{\tau}(b,\alpha) - \frac{\tau(b,\alpha)}{a}\right].$$

A solution requires $\tilde{V}(b,q) = q$, $x(b) = -\frac{\tau(b,\alpha)}{a}$. Substituting we get equation (25).

A.12 Proof of Proposition 5 and Corollary 6

1. Proof of Proposition 5

   By contradiction, suppose $q(b(T^P)|\alpha) > q^P(b(T^P))$. The government would be better off keep borrowing from the market and delay the intervention. This way it can relax its budget constraint: by borrowing at an higher price, it can maintain the same $\dot{b}$ but consume more. Suppose instead that $q(b(T^P)|\alpha) > q^P(b(T^P))$. Then, for a symmetric argument, the government would had been better off to anticipate the intervention.

2. Proof of Corollary 6

   $q^P(b)$ is decreasing in b, but independent from α. On the other hand, by (25), the market price $q(b|\alpha)$ is increasing in b and decreasing in α. It follows that $b(T^P)$ must be increasing in α and $q(b(T^P),\alpha)$must be decreasing in α.

A.13 Proof of Proposition 7

1. Part 1. It follows directly from the fact that $G(b) > 0$, and for $b(0) = 0$, the first term in (26) vanishes.

2. Part 2.

   Taking derivative w.r.t. α of equation (26), we have that

$$\frac{\partial}{\partial \alpha}\Delta V(\alpha, 0) = -(1-\alpha)\frac{\partial}{\partial \alpha}\int_{T^P}^{T^E} G(b(s), \alpha)e^{-(r+\lambda)T^E} ds + \int_{T^P}^{T^E} G(b(s), \alpha)e^{-(r+\lambda)T^E} ds$$

   Since $G(t)$ is positive $\forall t \geq T^P$, the second term is positive. For the first term to be positive for $\alpha > 1$ we need to show that

$$\frac{\partial}{\partial \alpha}\int_{T^P}^{T^E} G(b(s), \alpha)e^{-(r+\lambda)T^E} ds > 0.$$

   We use a perturbation argument. Suppose that we start from an equilibrium, where $b(T^P|\alpha)$ is debt at intervention and $b(T^E)$ is debt at default. First, notice that $b(T^E)$ is set by the policy-maker independently of α (the ICC does not depend on the fiscal rule). On the other hand, by altering the bond price, α affects the equilibrium debt at intervention. Consider a marginal increase in α, keeping intervention fixed at the initial equilibrium $b(T^P|\alpha)$. Increasing α shifts down the market bond price in (25), hence $q(b(T^P|\alpha)|\alpha + d\alpha) < q^P(b(T^P)|\alpha)$, $\forall d\alpha > 0$. By proposition 5, it cannot be optimal for the government to stop at $b(T^P|\alpha)$, better to stop one instant before. Therefore, it must be $b(T^P|\alpha + d\alpha) > b(T^P|\alpha)$ and $(T^E - T^P|\alpha + d\alpha) > (T^E - T^P|\alpha)$. Also, $G(b(t), \alpha)$ is decreasing in α since, for any given $\dot{b}(b(t))$, the distance $q^P(b(t)) - q(b(t)|\alpha + d\alpha) > q^P(b(t)) - q(b(t)|\alpha)$ is increasing $\forall b(t) \in [b(T^P|\alpha), b(T^E)]$. It follows that the derivative has a positive sign.

3. Part 3.

   For any given $b(T^P)$ and $d\alpha > 0$, $q(b(T^P)|\alpha) > q(b(T^P)|\alpha + d\alpha)$. This implies that an higher α restricts the inter-temporal budget constraint of the government as $\forall b < b(T^P)$, given the dynamic equation that characterizes the bond price dynamic before intervention in (25), it must be $q(b|\alpha) > q(b|\alpha + d\alpha)$. Indeed, to sustain any given borrowing plan $\{\dot{b}(t)\}_{t=0}^{T^P}$, the government will have to consume less. It follows that the welfare of the government should be monotonically decreasing in α.

A.14 Proof of Proposition 8

1. Let $\bar{\alpha}$ be such that $q(b(T^P)|\bar{\alpha}) = q(T)$. By the terminal conditions of the government problem, it must be $b(T^P|\bar{\alpha}) = b(T)$. It follows that the bond price pre-intervention is identical with and

without policy, hence $\Delta W(\bar{\alpha}, 0) = 0$. Since, by proposition (4) the intervention has net positive present value for investors, it must be $\Delta V(\bar{\alpha}, 0) > 0$. Equation 26, implies $\bar{\alpha} > 1$.

2. Let $\alpha = 1$, by equation (26) it follows immediately that $\Delta V(1, 0) = 0$. Moreover $\Delta W(1, 0) > 0$ since, by proposition 7, $\Delta W(\alpha, 0)$ is monotonically decreasing in α and from above we know that $\Delta W(\bar{\alpha}, 0) = 0$ for $\bar{\alpha} > 1$.

3. By proposition (7) it follows immediately that the Pareto set is identified by $\alpha \in [1, \bar{\alpha}]$.