

A Microfoundation for Normalized CES Production Functions with Factor–Augmenting Technical Change

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Abstract

We derive the aggregate normalized CES production function from idea-based micro-foundations where firms are allowed to choose their capital- and labor-augmenting technology optimally from a menu of available technologies. This menu is in turn augmented through factor-specific R&D. The considered model yields a number of interesting results. First, normalization of the production function can be maintained simultaneously at the local and at the aggregate level, greatly facilitating interpretation of the aggregate production function's parameters in terms of the underlying idea distributions. Second, in line with earlier findings, if capital- and labor-augmenting ideas are independently Weibull-distributed then the aggregate production function is CES; if they are independently Pareto-distributed, then it is Cobb–Douglas. Third, by disentangling technology choice by firms from R&D output, one can draw a clear-cut distinction between the direction of R&D and the direction of technical change actually observed in the economy, which are distinct concepts. Fourth, it is argued that the Weibull distribution should be a good approximation of the true unit factor productivity distribution (and thus the CES should be a good approximation of the true aggregate production function) if a “technology” is in fact an assembly of a large number of complementary components. This argument is illustrated with a novel, tractable model of directed (factor-specific) R&D. Finally, it is shown that all our results carry forward to the general case of n -input production functions.

Keywords: CES production function, normalization, Weibull distribution, direction of technical change, directed R&D, optimal technology choice

JEL Classification Numbers: E23, E25, O47

1 Introduction

Aggregate production functions – particularly Cobb–Douglas and CES functions – are used in virtually every paper in contemporary theoretical macroeconomics. Surprisingly few models have been put forward so far, however, in which these functions are derived from microfoundations. The purpose of the current contribution is to enrich this sparse literature with an analytically tractable microeconomic framework able to generate either of these two aggregative specifications endogenously. It is going to be an idea-based “endogenous technology choice” model where the aggregate production function is derived as a convex hull of local production functions (LPFs), chosen optimally by profit-maximizing firms. Each of these local techniques is in turn characterized by a pair of technology-specific unit factor productivities (UFPs), (a, b) , which augment labor and capital, respectively.

Similar frameworks have been studied in recent works by Jones (2005) and Growiec (2008a, 2008b).¹ Jones’ (2005) approach was based on the assumption that firms producing the final good draw capital- and labor-augmenting UFPs randomly from a pair of independent Pareto distributions, so that their technology choice is optimal only on average, or in the limit when sufficiently many draws have been made. Growiec (2008a) rewrote Jones’ model into a more tractable form that yields equivalent results. It enables the firms to pick their preferred technology pair (a, b) deterministically from a given technology menu and shifts the stochastic technology-search process to the R&D sector, composed of a continuum of researchers who draw the (a, b) technology pairs from a certain pre-defined joint bivariate distribution – constructed either from a pair of marginal Pareto distributions dependent according to the Clayton copula (Growiec, 2008a) or a pair of independent Weibull distributions (Growiec, 2008b). In all versions of this idea-based model, the shape of the resultant aggregate production function, obtained by plugging the optimal technology choices into LPFs, is found to depend, in general, both on the assumed shapes of LPFs and on the assumed joint distribution of UFPs.

Based on the above assumptions, these three papers have succeeded in providing idea-based microfoundations for Cobb–Douglas and CES aggregate production functions. Jones (2005) has shown that if capital- and labor-augmenting ideas are

¹Caselli and Coleman’s (2006) contribution is also related, but not as closely.

independently Pareto-distributed, then the aggregate production function is Cobb–Douglas; Growiec (2008a,b) has extended this result by proving that if they are independently Weibull-distributed, or Pareto-distributed and dependent according to the Clayton copula, then the aggregate production function is CES. These papers have overlooked a few important implications of the considered framework, though, most likely because of their somewhat cumbersome parametrization and a number of unnecessary implicit assumptions. The current article identifies several gaps in these papers and fills them, ultimately indicating that the “endogenous technology choice” model discussed there, once properly parametrized and relieved of unnecessary restrictions, has much more interesting features than it was uncovered so far. It provides an even more sound justification for the use of (normalized) CES production functions in macroeconomics.

Compared to the aforementioned contributions by Jones (2005) and Growiec (2008a, 2008b), the current paper makes four decisive changes in the model. Each of them is a source of a distinct contribution to the literature.

First, the model is rewritten in terms of *normalized* CES functions here (cf. La Grandville, 1989; Klump and La Grandville, 2000). Thanks to this step, we are now able to obtain an interpretable link between the parameters of the microfounded aggregate production function and the bivariate UFP distribution. The reason is that under normalization, CES production functions’ parameters represent separate concepts which are otherwise deeply intertwined: e.g., the distribution parameters of the un-normalized CES function are themselves functions of the elasticity of substitution and the normalized volume units (cf. Klump and Preissler, 2000).² We find that normalization with respect to initial inputs K_0, L_0 , output Y_0 and the initial capital income share π_0 , can be maintained simultaneously at the local and at the aggregate level, greatly facilitating the interpretation of the aggregate CES production function’s parameters in terms of the underlying idea distributions.

²This in turn greatly obstructs estimation of these two parameters and comparative statics exercises. A thorough elaboration of these issues as well as a survey of the related literature can be found in Klump et al. (2011). These authors also request that the parameters of production functions derived from microfoundations by Jones (2005) and Growiec (2008a, 2008b), should be provided with an interpretation consistent with normalization. Among other accomplishments, the current paper addresses this request.

Second, the current paper relaxes the assumption implicitly made in Growiec (2008a), that technological progress always augments the technology menu *proportionally*, as a homothetic transformation from the origin.³ In fact, this assumption can be easily generalized, allowing for directed factor-augmenting R&D (cf. Acemoglu, 2003),⁴ able to expand the technology menu in selected directions more than in others. This overturns some of the sharp predictions on the direction of technical change, put forward by Growiec (2008a). It also helps understand an important distinction, not mentioned in the earlier contributions: the *direction of R&D* (i.e., direction of expansion of the technology menu) is a distinct concept from the *direction of technical change* actually observed in an economy, because the evolution of firms' technology choices over time may not mirror the direction of augmentation of the technology menu. In particular, if the aggregate production function is CES with factor-augmenting technical change, these two concepts will be equivalent only along the balanced growth path in the unique case when it exists, that is when both R&D and technical change are purely labor-augmenting and when factor income shares are constant across time. This requires highly specific assumptions on the R&D process. Otherwise, the directions of R&D and technical change must diverge. In the Cobb–Douglas case, in turn, the direction of R&D does not have any impact on the direction of technical change: labor-augmenting technical change always follows changes in output per worker y , and capital-augmenting technical change always reflects changes in output per unit of capital, y/k .

Third, as opposed to the earlier contributions, the current model features also a novel, very detailed specification of the R&D sector. On its basis we construct an argument in support of the use of Weibull distributions in the context of R&D productivity (first suggested by Growiec, 2008b, but without any justification). It turns out that if factor-augmenting technologies are inherently complex and consist of a large number of complementary components, then the Weibull distribution should approximate the true productivity distribution better than any anything else, including the

³This assumption is also maintained by Jones (2005), but due to the multiplicative character of the Cobb–Douglas production function considered there, lifting this restriction does not change any results.

⁴There is a voluminous literature based on Acemoglu's (2003) framework. Important extensions have been put forward, among others, in Acemoglu (2007) and Acemoglu and Guerrieri (2008).

celebrated Pareto distribution (see Kortum, 1997; Gabaix, 1999; Jones, 2005, and references therein). The argument is based on the extreme value property of the Weibull distribution (cf. Kotz and Nadarajah, 2000; de Haan and Ferreira, 2006): if one takes n independent draws from some distribution that is bounded from below and satisfies an additional technical assumption (e.g., Pareto, uniform, truncated Gaussian, etc.) and takes the *minimum* of these draws, then as $n \rightarrow \infty$, this minimum will, after an appropriate normalization, converge to the standard Weibull distribution with a shape parameter $\alpha > 0$, dependent on the shape of the underlying sampling distribution. Clearly, taking the minimum applies to the case of complex technologies consisting of a number of complementary components (cf. Kremer, 1993; Blanchard and Kremer, 1997; Jones, 2011), because their productivity is then determined by the productivity of their “weakest link”. The importance of this finding for the current paper is corroborated by the fact that in the analyzed model, independent Weibull distributions give rise to the aggregate normalized CES production function.

Fourth, the current paper also demonstrates that the considered model is readily generalizable to n -factor production functions. Since all the derivations in the n -dimensional case are very similar to the two-dimensional case, formal elaboration of this issue has been delegated to the appendix.

The remainder of the paper is structured as follows. Section 2 derives the normalized CES production function from idea-based microfoundations. Section 3 shows how the static technology-choice framework can be embedded in a dynamic growth model and discusses the implications for the direction of technical change. Section 4 deals with the Cobb–Douglas case, demonstrating the key differences in comparison to the normalized CES case. Section 5 discusses the details of the model of directed R&D and discusses the conditions under which UFPs should be approximately Weibull-distributed. Section 6 concludes. The derivations in the general case of n -factor production functions have been delegated to the appendix.

2 Microfoundations for the aggregate normalized CES production function

In the current section, we shall show how to obtain the aggregate normalized CES production function from microfoundations – as a convex hull of LPFs, computed under the restriction that UFPs must be chosen from the given technology menu. The assumption regarding the postulated shape of this menu will be subsequently modified in Section 4 (where the aggregate Cobb–Douglas production function is derived), whereas in Section 5 it will be motivated with a more involved model of directed R&D and presented as a proposition.

2.1 Framework

The “endogenous technology choice” framework is based on the following assumptions.

Assumption 1 *The local production function (LPF) takes either the normalized CES or the normalized Leontief form:*

$$Y = \begin{cases} Y_0 \left(\pi_0 \left(\frac{bK}{b_0 K_0} \right)^\theta + (1 - \pi_0) \left(\frac{aL}{a_0 L_0} \right)^\theta \right)^{\frac{1}{\theta}}, & \text{if } \sigma_{LPF} \in (0, 1), \\ Y_0 \min \left\{ \left(\frac{bK}{b_0 K_0} \right), \left(\frac{aL}{a_0 L_0} \right) \right\}, & \text{if } \sigma_{LPF} = 0, \end{cases} \quad (1)$$

where $\theta \in [-\infty, 0)$ is the substitutability parameter, related to the elasticity of substitution via $\sigma_{LPF} = \frac{1}{1-\theta}$. Leontief LPFs, with $\sigma_{LPF} = 0$, are obtained as a special case of the more general normalized CES class of LPFs by taking the limit $\theta \rightarrow -\infty$ (we denote this case as $\theta = -\infty$ for simplicity). $\pi_0 = \frac{r_0 K_0}{Y_0}$ is the capital income share at t_0 . The LPF exhibits constant returns to scale.

Please note that in the normalization procedure, benchmark values have been assigned not only to output, capital and labor (Y_0, K_0, L_0), but also to the benchmark technology (b_0, a_0). In the following derivations, this benchmark technology will be identified with the *optimal* technology at time t_0 .

By assuming $\theta < 0$, or equivalently $\sigma_{LPF} < 1$, we concentrate on the likely case where LPFs allow little substitutability between inputs. More precisely, capital and labor are assumed to be gross complements in the LPF. All results derived in

this paper go through also in the limiting case of Leontief LPFs, where inputs are fully complementary. The CES specification of the LPF is thus not necessary for an aggregate CES production function to obtain.

The CES/Leontief specification of the LPF was used by Growiec (2008a,b), but without normalization.

Assumption 2 *The technology menu, specified in the (a, b) space, is given by equality:*

$$H(a, b) = \left(\frac{a}{\lambda_a}\right)^\alpha + \left(\frac{b}{\lambda_b}\right)^\alpha = N, \quad \lambda_a, \lambda_b, \alpha, N > 0. \quad (2)$$

In what follows, the technology menu will be understood as a contour line of the cumulative distribution function of the joint bivariate distribution of capital-augmenting ideas b and labor-augmenting ideas a . A formal justification for this assumption will be provided in Section 5. The key point there will be to obtain the technology menu from the individual (marginal) distributions of b and a . Under independence of both dimensions (so that marginal distributions are simply multiplied by one another), equation (2) obtains if and only if the marginal distributions are Weibull with the same shape parameter $\alpha > 0$ (Growiec, 2008b):⁵

$$P(\tilde{a} > a) = e^{-\left(\frac{a}{\lambda_a}\right)^\alpha}, \quad P(\tilde{b} > b) = e^{-\left(\frac{b}{\lambda_b}\right)^\alpha}$$

for $a, b > 0$. Under such parametrization, we have $P(\tilde{a} > a, \tilde{b} > b) = e^{-\left(\frac{a}{\lambda_a}\right)^\alpha - \left(\frac{b}{\lambda_b}\right)^\alpha}$, and thus the parameter N in eq. (2) is interpreted as $N = -\ln P(\tilde{a} > a, \tilde{b} > b) > 1$. In what follows, we will assume N to be constant across time, and λ_a, λ_b to grow as an outcome of factor-augmenting R&D.⁶

The analytical form of the technology menu postulated in equation (2) has also been used by Caselli and Coleman (2006) and Growiec (2008a,b), but with the unnecessary restriction that λ_a and λ_b are always shifted proportionately.

⁵Equation (2) can also be obtained for Pareto distributions of a and b , provided that the pattern of dependence between both marginal distributions is modeled with the Clayton copula (Growiec, 2008a). Its functional form has been first postulated by Caselli and Coleman (2006), but without any justification.

⁶One could easily reparametrize the technology menu, though, fixing either λ_a or λ_b and allowing N to vary. This would also reparametrize the resultant aggregate production function: the ratio N/N_0 would appear in equation (7) and the fixed parameter (λ_a or λ_b) would drop out. One could also (redundantly) vary all three parameters simultaneously. See the discussion in Section 3.

The case where a and b are independently Pareto distributed leads to a different specification of the technology menu which will be considered separately in Section 4. An important caveat is that if they are Weibull distributed but dependent, or independent but following some other distribution than Pareto or Weibull, the resultant aggregate production does not belong to the CES class. Such cases will not be considered here.

It is also vital that both marginal Weibull distributions share the same shape parameter α : if labor- and capital-augmenting ideas are independently Weibull distributed, but with different shape parameters, then the resultant aggregate production function does not belong to the CES class either. Fortunately, as Section 5 shows, under arguably general conditions the model of directed R&D discussed there will yield the same value of $\alpha = 1$.

Assumption 3 *Firms choose the technology pair (a, b) optimally, subject to the current technology menu, such that their profit is maximized:*

$$\max_{a,b} \left\{ Y_0 \left(\pi_0 \left(\frac{bK}{b_0 K_0} \right)^\theta + (1 - \pi_0) \left(\frac{aL}{a_0 L_0} \right)^\theta \right)^{\frac{1}{\theta}} \right\} \quad s.t. \quad \left(\frac{a}{\lambda_a} \right)^\alpha + \left(\frac{b}{\lambda_b} \right)^\alpha = N. \quad (3)$$

We note that factor remuneration $rK + wL$, taken account in the firms' profit maximization problem, does not depend on the chosen technology pair (a, b) so it can be safely omitted from the above optimization problem.⁷ The same assumption was made by Jones (2005) and Growiec (2008a,b).

Finally, second order conditions require us to assume that $\alpha > \theta$, so that the interior stationary point of the above problem is a maximum. The proof of this is included in the appendix. Furthermore, we also need to assume that $\alpha - \theta - \alpha\theta > 0$ so that the resultant aggregate production function is concave with respect to K and L . Both these conditions are satisfied automatically in the case $\alpha > 0 > \theta$, on which we concentrate here. As we shall see shortly, in such case, capital and labor are gross complements in the aggregate production function.

⁷In the case of Leontief LPFs, optimization requires $\frac{bK}{b_0 K_0} = \frac{aL}{a_0 L_0}$.

2.2 Technology choice and the aggregation result

Solving the maximization problem set up above yields direct results on the firm's optimal technology choices. First, at time t_0 , when $K = K_0, L = L_0, Y = Y_0, \lambda_a = \lambda_{a0}, \lambda_b = \lambda_{b0}$ is assumed, it is easily verified that the optimal choice is:

$$a_0^* = (N(1 - \pi_0))^{\frac{1}{\alpha}} \lambda_{a0}, \quad b_0^* = (N\pi_0)^{\frac{1}{\alpha}} \lambda_{b0}, \quad (4)$$

where λ_{a0} and λ_{b0} are the values of λ_a and λ_b at time t_0 , respectively. Values of a_0^* and b_0^* will be used as a_0 and b_0 in the normalization at the local level in all subsequent derivations.

For any other moment in time, the optimal technology choices are:

$$\left(\frac{a}{a_0}\right)^* = \frac{\lambda_a}{\lambda_{a0}} \left(\pi_0 \left(\frac{\lambda_b \lambda_{a0} K L_0}{\lambda_a \lambda_{b0} L K_0} \right)^{\frac{\alpha\theta}{\alpha-\theta}} + 1 - \pi_0 \right)^{-\frac{1}{\alpha}}, \quad (5)$$

$$\left(\frac{b}{b_0}\right)^* = \frac{\lambda_b}{\lambda_{b0}} \left(\pi_0 + (1 - \pi_0) \left(\frac{\lambda_b \lambda_{a0} K L_0}{\lambda_a \lambda_{b0} L K_0} \right)^{-\frac{\alpha\theta}{\alpha-\theta}} \right)^{-\frac{1}{\alpha}}, \quad (6)$$

where $\frac{\alpha\theta}{\alpha-\theta}$ is substituted with $-\alpha$ in the case of Leontief LPFs ($\theta = -\infty$).

Inserting these optimal technology choices into the LPF, we obtain the following result.

Proposition 1 *If Assumptions 1-3 hold, then the aggregate production function takes the normalized CES form:*

$$Y = Y_0 \left(\pi_0 \left(\frac{\lambda_b K}{\lambda_{b0} K_0} \right)^{\frac{\alpha\theta}{\alpha-\theta}} + (1 - \pi_0) \left(\frac{\lambda_a L}{\lambda_{a0} L_0} \right)^{\frac{\alpha\theta}{\alpha-\theta}} \right)^{\frac{\alpha-\theta}{\alpha\theta}}. \quad (7)$$

Hence, the normalized CES result obtains both in the case of CES and Leontief LPFs.

Proof (and generalization to n inputs): see the appendix. ■

It is worthwhile to comment on each of the parameters of the aggregate production function, because they all have sound interpretations:

- the substitutability parameter is $\rho = \frac{\alpha\theta}{\alpha-\theta}$ (or $\rho = -\alpha$ in the case of Leontief LPFs). The elasticity of substitution is thus $\sigma = \frac{1}{1-\rho} = \frac{\alpha-\theta}{\alpha-\theta-\alpha\theta} > 0$ (or $\sigma = \frac{1}{1+\alpha}$ in the case of Leontief LPFs). It is verified that $\theta < \rho < 0$ and thus $\sigma_{LPF} = \frac{1}{1-\theta} < \sigma < 1$. Hence, endogenous technology choice *unambiguously increases*

the substitutability between production factors as compared to the LPF, but this substitutability nevertheless remains bounded from above by the unitary elasticity of substitution, characteristic for the Cobb–Douglas specification,

- the distribution parameter is $\pi_0 = \frac{r_0 K_0}{Y_0}$, whereas the multiplicative constant term is Y_0 . Hence, thanks to normalization, both these parameters are equal to the respective parameters of the LPF;
- the constant parameter N does not appear in the aggregate production function,
- the capital-augmenting factor b present in the LPF is replaced by the capital-augmenting parameter of the technology menu λ_b in the aggregate production function, both at time t_0 and at the current time. The same applies to labor-augmenting factor a and the respective parameter λ_a .⁸

Hence, all growth in λ_a and λ_b (with N kept intact), obtained thanks to directed R&D, will ultimately appear as a multiplicative term in front of the respective factor of production in the aggregate production function. As it will be shown in the following section, however, growth in λ_a or λ_b ought not to be confused with the actual factor-augmenting technical change, that is growth in a and b : already from equations (5)–(6), one sees that these two types of entities are generally not proportional to one another, unless additional conditions are met.

We also note the following straightforward corollary.

Corollary 1 *Assuming that factors are priced at their marginal product, the capital and labor income shares are equal to, respectively:*

$$\pi = \frac{rK}{Y} = \frac{\pi_0 \left(\frac{\lambda_b}{\lambda_{b0}} \frac{K}{K_0} \right)^{\frac{\alpha\theta}{\alpha-\theta}}}{\pi_0 \left(\frac{\lambda_b}{\lambda_{b0}} \frac{K}{K_0} \right)^{\frac{\alpha\theta}{\alpha-\theta}} + (1 - \pi_0) \left(\frac{\lambda_a}{\lambda_{a0}} \frac{L}{L_0} \right)^{\frac{\alpha\theta}{\alpha-\theta}}}, \quad (8)$$

$$1 - \pi = \frac{wL}{Y} = \frac{(1 - \pi_0) \left(\frac{\lambda_a}{\lambda_{a0}} \frac{L}{L_0} \right)^{\frac{\alpha\theta}{\alpha-\theta}}}{\pi_0 \left(\frac{\lambda_b}{\lambda_{b0}} \frac{K}{K_0} \right)^{\frac{\alpha\theta}{\alpha-\theta}} + (1 - \pi_0) \left(\frac{\lambda_a}{\lambda_{a0}} \frac{L}{L_0} \right)^{\frac{\alpha\theta}{\alpha-\theta}}}. \quad (9)$$

⁸The last two findings are a corollary from the fact that the technology menu defined in Assumption 2 is a curve in a two-dimensional space, parametrized by three parameters λ_a, λ_b, N , and N is kept constant.

The above factor share formulas are instructive as regards the expected direction of their change: this direction is strictly determined by the growth rate of $\lambda_b K$ relative to $\lambda_a L$. If both growth rates are equal, the capital income share will remain constant at π_0 . If $\lambda_b K$ grows faster, then due to gross complementarity ($\frac{\alpha\theta}{\alpha-\theta} < 0$), capital's income share will gradually fall to zero over time; conversely, it will gradually rise towards unity if $\lambda_a L$ grows faster. Hence, endogenous technology choice does not overturn any of the standard results identified in earlier literature.⁹

⁹An obvious remark here is that the above factor share formulas are invalidated once one allows for imperfect competition. However, if it is introduced via the Dixit–Stiglitz model of monopolistic competition, implying constant markups over marginal costs, then this change would merely re-scale factor income shares, without altering any of the results on their dynamics.

3 Implications for the direction of technical change

Given the static character of the endogenous technology choice framework, it is straightforward to embed it in dynamic growth models. In particular, one could assume additionally that there are no interactions between technology choice and factor demand on the side of firms. In such case, the aggregate production function derived in equation (7) would enter the dynamic model directly, and the model could be closed, e.g., by allowing directed R&D to increase λ_a and λ_b endogenously (cf. Acemoglu, 2003). Then, depending on the assumptions of the embedding growth model, generally any direction of R&D could be obtained as an equilibrium.¹⁰

In the following subsections, we shall discuss a few examples of growth models that can be used as such embedding structures.

3.1 Balanced growth path, Harrod–neutral R&D, and purely-augmenting technical change

It is very often assumed in the economic growth literature – partly due to some empirical “stylized facts”, and partly due to analytical convenience – that the economy follows a balanced growth path, or at least converges to it. It should be emphasized, however, that such an assumption is a very stringent, knife-edge one (cf. Growiec, 2007): it requires that either the aggregate production function is Cobb–Douglas or technical change is purely labor-augmenting (Uzawa, 1961).

A seminal example of a directed R&D setup that gives rise to a balanced growth path in the case of a CES production function with purely labor-augmenting technical change is due to Acemoglu (2003). It can be straightforwardly used as an embedding structure over our technology choice framework. The production functions for factor-augmenting shifts in the technology menu are then captured by linear equations of form:

$$\dot{\lambda}_a = f_a(\ell_a)\lambda_a, \tag{10}$$

$$\dot{\lambda}_b = f_b(\ell_b)\lambda_b, \tag{11}$$

where ℓ_a and ℓ_b are the fractions of population engaged in labor- and capital-augmen-

¹⁰Albeit perhaps only some of them would be useful for an economist.

ting R&D, respectively, and f_a, f_b are some smooth increasing functions. Such a model is scale-free. An analysis of the model's implications, both in the social planner allocation and the decentralized equilibrium à la Acemoglu, reveals that the economy converges to a balanced growth path where:¹¹

$$\hat{y} = \hat{k} = \hat{a} = \hat{\lambda}_a = f(\ell_a^*), \quad (12)$$

$$\hat{b} = \hat{\lambda}_b = 0, \quad (13)$$

and thus (i) R&D is Harrod-neutral, i.e., directed toward labor-augmenting developments only ($\hat{\lambda}_b = 0$), (ii) technical change is also Harrod-neutral, i.e., purely labor-augmenting ($\hat{b} = 0$), (iii) factor income shares are constant at π_0 and $1 - \pi_0$, respectively, despite the fact that the production function is not Cobb-Douglas.

Hence, in this very specific framework with linear technology equations in both R&D sectors and no intrasectoral spillovers between both sectors, technical change must follow the direction of R&D on one-to-one basis.

Further analysis shows that the capital- and labor-augmenting idea production functions used in this directed R&D model can be made slightly more general without altering any of the above predictions. This would happen if one allowed mutual spillovers between both R&D sectors, yet also imposed a particular knife-edge condition on their strength (measured by partial elasticities). This result has been obtained by Li (2000), for a somewhat different two-R&D-sector model, which is however identical to the current one in its reduced form (that is, after stripping its solution to the form of a system of dynamic equations governing the dynamics of its variables).

What is much more important here, however, is that for *every other reduced-form specification* of the growth model, the above balanced-growth-path result must fail. Hence in the *typical* (non-knife-edge) case, technical change will *not* be purely labor-augmenting, it will *not* reflect the direction of R&D, and factor income shares will *not* be constant across time.

3.2 Hicks-neutral R&D

Let us now pass to another specific case, the Hicks-neutral one implying that R&D expands the technology menu proportionally, so that it is not biased towards any

¹¹We use the notation: $y \equiv Y/L, k \equiv K/L$, and $\hat{x} \equiv \dot{x}/x$ for any variable x .

of the sectors. Though it is as parsimonious as the Harrod–neutral case discussed above – obtaining this particular case also requires one to make a certain knife-edge assumption – it provides us with another benchmark case to which we can compare the results of all non-neutral cases.

Hicks–neutral R&D has been considered in a very similar endogenous technology choice framework by Caselli and Coleman (2006) and Growiec (2008a). The implicit assumption made there, that factor-augmenting idea distributions \tilde{a} and \tilde{b} evolve proportionally, is equivalent to positing that the ratio λ_a/λ_b is constant, and thus $\hat{\lambda}_a = \hat{\lambda}_b$ for all times t .¹² Again, one obtains such a result only in a certain knife-edge case. For example, R&D is Hicks–neutral for all t in a dynamic model where

$$\dot{\lambda}_a = f(\ell_a, \ell_b) \lambda_a^{\alpha+1} \lambda_b^\beta, \quad (14)$$

$$\dot{\lambda}_b = f(\ell_a, \ell_b) \lambda_a^\alpha \lambda_b^{\beta+1}, \quad (15)$$

and hence by assumption $\hat{\lambda}_a = \hat{\lambda}_b$.¹³

As argued above, Hicks–neutral R&D precludes the existence of a balanced growth path. More surprisingly, however, it also implies that technical change in the aggregate economy, determined jointly by the direction of R&D and firms’ endogenous technology choices, is *not* Hicks–neutral. In particular, under the assumptions that (i) Hicks–neutral R&D improves UFPs at a constant rate, so that $\hat{\lambda}_a = \hat{\lambda}_b \equiv g > 0$, (ii) the economy is able to maintain positive growth rates of physical capital per worker k until infinite time, with $\lim_{t \rightarrow \infty} k(t) = +\infty$, technical change will augment both factors of production in the long run, according to:

$$\lim_{t \rightarrow \infty} \hat{y}(t) = g, \quad (16)$$

$$\lim_{t \rightarrow \infty} \hat{a}(t) = g, \quad (17)$$

$$\lim_{t \rightarrow \infty} \hat{b}(t) = g + \frac{\theta}{\alpha - \theta} \lim_{t \rightarrow \infty} \hat{k}(t) \Rightarrow \lim_{t \rightarrow \infty} \hat{b}(t) \in \left[\left(\frac{\alpha}{\alpha - \theta} \right) g, g \right). \quad (18)$$

¹²Growiec (2008a) assumed that λ_a and λ_b were fixed, and proxied technological progress by growth in N instead. This is equivalent, however, in terms of the evolution of the technology menu over time, to keeping N fixed and varying λ_a and λ_b proportionately. An analogous assumption was made by Caselli and Coleman (2006) in the cross-sectional context: they allowed only N to vary across countries, but λ_a and λ_b were kept fixed.

¹³In a somewhat larger (yet, still very specific) class of models, R&D will be Hicks–neutral in the limit of $t \rightarrow \infty$. The long-run results obtain within this section hold for such models as well.

These results have been obtained by taking limits of optimal technology choices (5)–(6) under the assumption that $k(t) \rightarrow +\infty$. We have also used the inequality $\lim_{t \rightarrow \infty} \hat{k}(t) \leq \lim_{t \rightarrow \infty} \hat{y}(t)$, because in the opposite case, $y(t)/k(t)$ would be falling towards zero, ultimately violating the capital’s equation of motion.¹⁴

Hence, technological change augments both factors of production in the long run. Capital-augmenting technical change remains positive forever, too, in contrast to the findings based on the Cobb–Douglas production function (Jones, 2005).

On the other hand, even if R&D is Hicks–neutral, endogenous technology choice introduces a bias in the direction of technical change, in favor of labor, i.e., the non-accumulable input. Firms decide optimally to increase the UFP of labor faster than that of capital in order to adjust to the ongoing changes in factor proportions, which are in favor of capital. This is natural given gross complementarity of both inputs.

The capital income share π is bound to fall gradually towards zero under Hicks–neutral R&D, provided that the rate of capital accumulation remains positive over the long run.

3.3 All other directions of R&D

For all other cases of directed R&D, implying $\hat{\lambda}_a \neq \hat{\lambda}_b$ and $\hat{\lambda}_b \neq 0$ over the long run, we obtain the following generic results.

- If $\lambda_b K$ grows faster than $\lambda_a L$, then $\hat{y}(t) \rightarrow \hat{\lambda}_a(t)$, $\hat{a}(t) \rightarrow \hat{\lambda}_a(t)$ and $\hat{b}(t) \rightarrow \hat{\lambda}_b(t) + \frac{\theta}{\alpha - \theta} \hat{k}(t)$ with t .¹⁵ The capital income share falls towards zero over time.
- If $\lambda_b K$ grows slower than $\lambda_a L$, then $\hat{y}(t) \rightarrow \hat{\lambda}_b(t) + \hat{k}(t)$, $\hat{a}(t) \rightarrow \hat{\lambda}_a(t) - \frac{\theta}{\alpha - \theta} \hat{k}(t)$ and $\hat{b}(t) \rightarrow \hat{\lambda}_b(t)$ with t .¹⁶ The capital income share increases towards unity over time.
- If $\lambda_b K$ grows asymptotically at the same pace as $\lambda_a L$, then $\hat{y}(t) \rightarrow \hat{\lambda}_a(t) =$

¹⁴Assuming furthermore that $\lim_{t \rightarrow \infty} \hat{k}(t) = \lim_{t \rightarrow \infty} \hat{y}(t) = g$, it follows that $\lim_{t \rightarrow \infty} \hat{b}(t) = \left(\frac{\alpha}{\alpha - \theta}\right) g > 0$.

¹⁵Given that $\hat{k}(t) \leq \hat{y}(t)$ for sufficiently large t , this case can only obtain if $\hat{\lambda}_b(t) > 0$.

¹⁶If $\hat{k}(t) = \hat{y}(t)$ in the long run and $\hat{\lambda}_b > 0$, then such a model would imply explosive dynamics, potentially achieving infinite output in finite time.

$\hat{\lambda}_b(t) + \hat{k}(t)$, $\hat{a}(t) \rightarrow \hat{\lambda}_a(t)$ and $\hat{b}(t) \rightarrow \hat{\lambda}_b(t)$ with t .¹⁷ The capital income share tends to a constant.

Hence, in the general case, the direction of R&D and the direction of technical change are different from one another. Optimal technology choice as well as the evolution of factor income shares are determined by comparing the growth rates of $\lambda_b K$ and $\lambda_a L$, i.e., of capital and labor in efficient units evaluated at the level of the aggregate production function.

¹⁷If $\hat{k}(t) = \hat{y}(t)$ in the long run, then this case can appear only if $\hat{\lambda}_b(t) \rightarrow 0$, which boils down to the balanced growth path case discussed above.

4 Cobb–Douglas aggregate production function

Let us now demonstrate how to derive the aggregate Cobb–Douglas production function within our framework (cf. Jones, 2005). The key change in assumptions that is required to produce this result relates to the distribution of capital- and labor-augmenting ideas; everything else is preserved. This said, in Section 5 we will argue why this change might be actually misleading, and why the aggregate CES production function might in fact be a more plausible alternative. We think however that the Cobb–Douglas case is a useful benchmark for comparisons because it is frequently used in the literature.

4.1 Modification of the framework

Let us now replace Assumption 2 with the following one:

Assumption 4 (modification of Assumption 2) *The technology menu, defined in the (a, b) space, is given by the equality:*

$$H(a, b) = \left(\frac{a}{\lambda_a}\right)^{\phi_L} \left(\frac{b}{\lambda_b}\right)^{\phi_K} = N, \quad \phi_K, \phi_L > 0. \quad (19)$$

This shape of the technology menu is consistent with the assumption that a and b are independently Pareto-distributed, with shape parameters ϕ_L and ϕ_K , respectively:

$$P(\tilde{a} > a) = \left(\frac{\lambda_a}{a}\right)^{\phi_L}, \quad P(\tilde{b} > b) = \left(\frac{\lambda_b}{b}\right)^{\phi_K}, \quad (20)$$

for $a > \lambda_a$ and $b > \lambda_b$. In such case, $N = \frac{1}{P(\tilde{a} > a, \tilde{b} > b)}$. The same assumption was made previously by Jones (2005), but with the unnecessary restriction of proportional (Hicks–neutral) augmentation of the technology menu.

4.2 Technology choice and the aggregation result

At t_0 , when $K = K_0, L = L_0, Y = Y_0, \lambda_a = \lambda_{a0}, \lambda_b = \lambda_{b0}$ is assumed, the optimal choice is indeterminate, provided that

$$\pi_0 = \frac{r_0 K_0}{Y_0} = \frac{\phi_K}{\phi_L + \phi_K}. \quad (21)$$

This restriction means that the capital income share at t_0 (and in fact at all other times as well) should be equal to $\frac{\phi_K}{\phi_L + \phi_K}$. Thus, π_0 ceases to be a free parameter, and a_0 becomes a free parameter instead (b_0 is then calculated according to $\left(\frac{a_0}{\lambda_{a0}}\right)^{\phi_L} \left(\frac{b_0}{\lambda_{b0}}\right)^{\phi_K} = N$).

At any other moment in time, and given a_0 and b_0 , the optimal technology choices are:

$$\left(\frac{a}{a_0}\right)^* = \left(\frac{\lambda_a}{\lambda_{a0}}\right)^{\frac{\phi_L}{\phi_L + \phi_K}} \left(\frac{\lambda_b}{\lambda_{b0}}\right)^{\frac{\phi_K}{\phi_L + \phi_K}} \left(\frac{KL_0}{LK_0}\right)^{\frac{\phi_K}{\phi_L + \phi_K}} \quad (22)$$

$$\left(\frac{b}{b_0}\right)^* = \left(\frac{\lambda_a}{\lambda_{a0}}\right)^{\frac{\phi_L}{\phi_L + \phi_K}} \left(\frac{\lambda_b}{\lambda_{b0}}\right)^{\frac{\phi_K}{\phi_L + \phi_K}} \left(\frac{KL_0}{LK_0}\right)^{-\frac{\phi_L}{\phi_L + \phi_K}}. \quad (23)$$

Inserting these optimal technology choices into the LPF, we obtain the following result.

Proposition 2 *If Assumptions 1, 3, and 4 hold, then the aggregate production function takes the Cobb–Douglas form:*

$$Y = Y_0 \left(\frac{\lambda_a}{\lambda_{a0}}\right)^{\frac{\phi_L}{\phi_L + \phi_K}} \left(\frac{\lambda_b}{\lambda_{b0}}\right)^{\frac{\phi_K}{\phi_L + \phi_K}} \left(\frac{K}{K_0}\right)^{\frac{\phi_K}{\phi_L + \phi_K}} \left(\frac{L}{L_0}\right)^{\frac{\phi_L}{\phi_L + \phi_K}}. \quad (24)$$

Proof (and generalization to n inputs): see the appendix. ■

The interpretation of the parameters of the aggregate Cobb–Douglas production function is the following:

- the distribution parameter is $\pi_0 = \frac{r_0 K_0}{Y_0} = \frac{\phi_K}{\phi_L + \phi_K}$. Under normalization, the distribution parameter of the LPF π_0 has to be assumed equal to the (constant) capital income share of the aggregate Cobb–Douglas production function,
- partial elasticities of capital and labor in the aggregate production function are proportional to the shape parameters of the Pareto distributions of their respective factor-augmenting technologies and sum up to one (guaranteeing constant returns to scale),
- the multiplicative constant term is Y_0 . Thanks to normalization, it is thus exactly equal to the multiplicative constant term of the LPF,

- the constant parameter N does not appear in the aggregate production function,¹⁸
- the capital-and labor-augmenting parameters of the technology menu, λ_b and λ_a respectively, enter the aggregate production function multiplicatively, taken to their respective powers ϕ_K and ϕ_L . Growth in aggregate output is thus invariant to the direction of R&D.

4.3 Direction of technical change

Under endogenous technology choice, the Cobb–Douglas case provides very specific implications for the direction of technical change. To see them, log-differentiate equations (22)–(24) with respect to time and compare terms to obtain:

$$\hat{a} = \hat{y} = \frac{\phi_L}{\phi_L + \phi_K} \hat{\lambda}_a + \frac{\phi_K}{\phi_L + \phi_K} \hat{\lambda}_b + \frac{\phi_K}{\phi_L + \phi_K} \hat{k}, \quad (25)$$

$$\hat{b} = \hat{y} - \hat{k} = \frac{\phi_L}{\phi_L + \phi_K} \hat{\lambda}_a + \frac{\phi_K}{\phi_L + \phi_K} \hat{\lambda}_b - \frac{\phi_L}{\phi_L + \phi_K} \hat{k}. \quad (26)$$

It follows that in the Cobb–Douglas case, no matter what the direction of R&D is, i.e., irrespective of the values of $\hat{\lambda}_a$ and $\hat{\lambda}_b$, firms will always adjust the labor-augmenting technology on one-to-one basis to changes in output per worker y , and capital-augmenting technology will be, accordingly, always adjusted one-to-one to changes in output per unit of capital y/k . Hence, as shown by Jones (2005), technological change must be purely labor-augmenting along the balanced growth path, where the output–capital ratio y/k is constant.

The capital income share is now fixed at $\pi_0 = \frac{\phi_K}{\phi_L + \phi_K}$, and the labor income share is fixed at $1 - \pi_0 = \frac{\phi_L}{\phi_L + \phi_K}$, for all times t .

¹⁸Again, one could easily reparametrize the technological menu, fixing either λ_a or λ_b and allowing N to vary across time. In such case, the ratio N/N_0 (which now drops out) would appear in equation (24).

5 The Weibull distribution in R&D productivity

After having discussed the key properties of the endogenous technology choice model where the aggregate production function is derived as a convex hull of local production techniques, with UFPs selected from the given technology menu, let us now justify the functional form of this menu, taken for granted in Assumption 2. This will be done using a novel, analytically tractable model of two independent R&D sectors, producing capital- and labor-augmenting innovations, respectively. It bears some similarity with the framework discussed in Appendix D of Growiec (2008a), but has a few unique distinguishing features. The model will be characterized in the two following subsections.

5.1 Distributions of complex ideas

The key novelty of the current model is the assumption that ideas are inherently *complex* and consist of a large number of *complementary* components. This is summarized in the following assumption.

Assumption 5 *The (capital- or labor-augmenting) R&D sector consists of an infinity of researchers located along the unit interval $I = [0, 1]$. At each instant t , every researcher $i \in I$ determines the quality of her innovation (\tilde{b}_i or \tilde{a}_i , respectively) by taking the minimum over n independent draws from the elementary idea distribution with cdf \mathcal{F} . The distribution \mathcal{F} has positive density on $[w, v)$, where v can be infinite, and zero density otherwise, and satisfies the condition*

$$\lim_{p \rightarrow 0^+} \frac{\mathcal{F}(w + px)}{\mathcal{F}(w + p)} = x^\alpha$$

for all $x > 0$ and a certain $\alpha > 0$.

The parameter n in the above assumption captures the number of constituent components of any given (composite) idea, and thus captures the complexity of any state-of-the-art technology. Allowing for such complexity puts the current framework in stark contrast to earlier studies (such as Jones, 2005, or Growiec, 2008a) where the

quality of ideas was determined via a *single* draw from the elementary idea distribution \mathcal{F} .¹⁹

The assumption that the quality of an innovation is the *minimum* (a Leontief function) of a range of n independent draws from the distribution \mathcal{F} reflects the view that the components of an idea are complementary to one another (Kremer, 1993; Blanchard and Kremer, 1997; Jones, 2011). More precisely, we consider the extreme case where they are *perfectly* complementary, and thus total productivity of a complex idea is determined by the productivity of its “weakest link” (or “bottleneck”). Clearly, this need not hold exactly in reality, since certain deficiencies of design can often be covered by advantages in different respects. However, the example of the explosion of the space shuttle *Challenger* due to a failure of an inexpensive O-ring, put forward by Kremer (1993), is perhaps the best possible illustration of the potentially complementary character of components of complex ideas.

Letting the technology complexity n be arbitrarily large, we obtain the following result:

Proposition 3 *If Assumption 5 holds, then as $n \rightarrow \infty$, the minimum of n independent random draws from the distribution with cdf \mathcal{F} , after appropriate normalization, converges in distribution to the Weibull distribution with the shape parameter α :*

$$[1 - \mathcal{F}(xp_n + w)]^n \xrightarrow{d} e^{-\left(\frac{x}{\lambda}\right)^\alpha}, \quad (27)$$

where $w = \inf\{x \in \mathbb{R} : \mathcal{F}(x) > 0\}$, $p_n = \frac{1}{\lambda} (\mathcal{F}^{-1}(\frac{1}{n}) - w)$ and the free parameter $\lambda > 0$ is assumed to be proportional to the mean of the underlying distribution \mathcal{F} .

Proof. The proposition follows directly from the Fisher–Tippett–Gnedenko extreme value theorem, applied to the distribution \mathcal{F} (Theorem 1.1.3 in de Haan and Ferreira, 2006, rephrased so that it captures the minimum instead of maximum). From the

¹⁹Jones (2005) viewed the technology menu as a convex hull of a finite number N of ideas. Hence, in the limit $N \rightarrow \infty$, this menu took the form of a contour line of a Fréchet distribution, which is the limiting distribution of the *maximum* of N independent draws from a distribution \mathcal{F} that is bounded from below. This assumption has been later replaced, both in Growiec (2008a) and in the current paper, with Assumption 6. At this point, one should note that Jones (2005) was preoccupied with the distribution of the maximum *across ideas* and here we are considering the minimum across *components of each idea*. Across ideas, it is still the best draws that matter.

theorem specifying the domain of attraction of the Weibull distribution (Theorem 1.2.1 in de Haan and Ferreira, 2006; Section 1.3 in Kotz and Nadarajah, 2000), we obtain the necessary and sufficient conditions guaranteeing for the complementarity mechanism to work. ■

From the mathematical point of view, the parameter λ is superfluous in the above derivations and can be normalized to unity by a simple re-normalization of the sequence p_n as $\tilde{p}_n = p_n \cdot \lambda = \mathcal{F}^{-1}\left(\frac{1}{n}\right) - w$. We think however that it is important to maintain the distinction between p_n and λ in order to maintain the ability of the R&D sector in the model to influence the means of capital- and labor-augmenting ideas. And it is precisely λ which pins down the mean of the resultant Weibull distribution.

Hence, when turning to our distinction between capital- and labor-augmenting R&D, we shall distinguish between λ_a determining the mean of \tilde{a} , and λ_b pinning down the mean of \tilde{b} , obtaining the following generic results:

$$E\tilde{a} = \lambda_a \Gamma\left(1 + \frac{1}{\alpha}\right), \quad E\tilde{b} = \lambda_b \Gamma\left(1 + \frac{1}{\alpha}\right), \quad (28)$$

where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ is the Euler's Gamma function.

A few meaningful applications of Proposition 3 have been summarized in Table 1. They indicate that the Weibull result can be obtained for a number of classes of distributions frequently discussed in the literature. They also provide a clear explanation how to relate λ to the characteristics of the underlying productivity distribution, and what values of the parameter α should be expected. Extreme value theory provides sharp implications on that.

The main message drawn from the results presented in Table 1 is that we can model directed R&D, affecting the mean of the underlying distribution \mathcal{F} and thus λ_a and λ_b , in an arbitrary way; if only the conditions specified in Assumption 5 hold, then the Weibull distribution result will always go through. If, on top of that, the shape parameters α of capital- and labor-augmenting developments happen to be equal to one another, then the aggregate normalized CES production function result will always follow, too.

Table 1: Selected distributions \mathcal{F} such that for $X_1, \dots, X_n \sim \mathcal{F}$, $\min\{X_1, \dots, X_n\}$ converges in distribution to the Weibull distribution as $n \rightarrow \infty$.

Distribution	Cdf for $x \in [w, v)$	Lower bound	Postulated p_n	Implied λ	Implied α
Pareto(ϕ)	$\mathcal{F}(x) = 1 - \left(\frac{\lambda x}{x}\right)^\phi$	$w = \gamma_x$	$p_n = \left(1 - \frac{1}{n}\right)^{-\frac{1}{\phi}}$	$\lambda = \gamma_x$	$\alpha = 1$
Uniform $U([r, s])$	$\mathcal{F}(x) = \frac{x-r}{s-r}$	$w = r$	$p_n = \frac{1}{n}$	$\lambda = s - r$	$\alpha = 1$
Truncated $N(\mu, \sigma)$	$\mathcal{F}(x) = \frac{\Phi\left(\frac{x-\mu}{\sigma}\right) - \Phi\left(\frac{w-\mu}{\sigma}\right)}{1 - \Phi\left(\frac{w-\mu}{\sigma}\right)}$	given w	$p_n = \bar{p}_n$	$\lambda = \mu$	$\alpha = 1$
Weibull(α, λ)	$\mathcal{F}(x) = 1 - e^{-\left(\frac{x}{\lambda}\right)^\alpha}$	$w = 0$	$p_n = -\frac{1}{\alpha} \ln\left(1 - \frac{1}{n}\right)$	given λ	given α

Notes: (i) to obtain convergence to the Weibull distribution, one may equivalently take $p_n = \frac{1}{n}$ in the Pareto case, and $p_n = n^{-\frac{1}{\alpha}}$ in the Weibull case;

(ii) we used the notation

$$\bar{p}_n = 1 - \frac{w}{\mu} + \frac{\sigma}{\mu} \Phi^{-1} \left(\left[1 - \Phi \left(\frac{w - \mu}{\sigma} \right) \right] \frac{1}{n} + \Phi \left(\frac{w - \mu}{\sigma} \right) \right).$$

The mean of a random variable drawn from the truncated Gaussian distribution increases both with the mean of the original distribution μ and the truncation point w , according to the formula $EX = \mu + \frac{\sigma \varphi\left(\frac{w-\mu}{\sigma}\right)}{1 - \Phi\left(\frac{w-\mu}{\sigma}\right)}$. We have chosen technological change in λ to affect μ , but we might have alternatively chosen it to affect w , or some mixture of both.

Fortunately, the implied parameter α is found to be unitary for a wide range of distributions \mathcal{F} , and therefore the condition that α is equal for both capital- and labor-augmenting ideas is quite plausible. Furthermore, if $\alpha = 1$, then also $E\tilde{a} = \lambda_a$ and $E\tilde{b} = \lambda_b$, which makes the link between the underlying distribution \mathcal{F} and the resultant Weibull distribution (which in such case specializes to the exponential distribution) especially apparent. We note the following corollary.

Corollary 2 *If the underlying idea distributions \mathcal{F} are Pareto, uniform or truncated Gaussian, then $\alpha = 1$ and thus the resultant idea distribution is exponential. In such case, the elasticity of substitution of the aggregate CES production function is equal to $\sigma = \frac{1-\theta}{1-2\theta} \in [\frac{1}{2}, 1)$, increasing from $\frac{1}{2}$ in the case of Leontief LPFs to unity in the limiting case of Cobb–Douglas LPFs.*

Please also note that a number of frequently considered classes of distributions have not been included in Table 1, because they do not satisfy the conditions of the theorem. First of all, the support of the distribution must be bounded from below, which rules out distributions defined on \mathbb{R} such as the Gaussian. Also, the pdf of such distribution cannot increase smoothly from zero at w ; there must be a jump. This rules a few more candidate distributions such as the lognormal or the Fréchet. Furthermore, the lowest possible value of the random variable cannot be an isolated atom, which rules out all discrete distributions such as the two-point distribution, the binomial, negative binomial, Poisson, etc.

5.2 Derivation of the technology menu

Let us now finally show how the individual draws of (complex, factor-augmenting) technologies \tilde{a} and \tilde{b} are combined, yielding the functional form of the technology menu postulated in Assumptions 2 and 4. We shall close the model of the R&D sector by making the following assumption.

Assumption 6 *Every capital- and labor-augmenting technology draw is allowed to enter the technology menu if it has been confirmed by at least a pre-defined fraction of researchers in I (z_b and z_a , respectively).*

Given the above assumption and the Law of Large Numbers, a labor-augmenting technology a can be included in the technology menu at time t if $P(\tilde{a} > a) \geq z_a$, and a capital-augmenting technology b , if $P(\tilde{b} > b) \geq z_b$. Since both R&D sectors are independent from one another, an aggregate technology (a, b) is thus included in the technology menu if $P(\tilde{a} > a, \tilde{b} > b) = P(\tilde{a} > a)P(\tilde{b} > b) \geq z_a z_b$. Since no profit-maximizing firm will choose a dominated technology, we may as well replace the above “ \geq ” inequality with equality in the formulation of the technology menu. This brings us directly to Assumption 2, if the distributions of \tilde{a} and \tilde{b} are Weibull, or to Assumption 4, if these distributions are Pareto. Moreover, as we have shown above, under the assumption that each idea consists of n complementary components, with $n \rightarrow \infty$, they should converge to the Weibull distribution and thus the resultant aggregate production should be CES.

6 Conclusion

The objective of the current paper has been to provide an idea-based microfoundation for a few selected shapes of aggregate production functions, commonly discussed in the literature. To this end, we have shown that the normalized CES production function can be derived as a convex hull of local production techniques under the assumption that labor- and capital-augmenting ideas are independently Weibull distributed (Growiec, 2008a,b). If they are independently Pareto distributed, then the resultant aggregate production function will be Cobb–Douglas (Jones, 2005). If neither of these options holds, then the resultant aggregate production function will not belong to the CES class.

This result has a number of interesting features. First, thanks to normalization, all parameters of the derived CES and Cobb–Douglas aggregate production functions have a sound interpretation in terms of the parameters of local production techniques and the underlying unit factor productivity distributions. Second, we find that the elasticity of substitution in the aggregate production function is unambiguously higher than in local production techniques, signifying that if the production function is viewed as an assembly of heterogeneous technologies, *technological substitution* can effectively augment factor substitution. Third, normalization can be maintained simultaneously at the local and aggregate level.

The next step taken in the current paper was to embed our static endogenous technology choice framework in a dynamic growth model. This step provided us with an opportunity to derive a number of theoretical predictions regarding the (endogenously determined) direction of technical change (labor-augmenting vs. capital-augmenting). Marked differences have been found here between the normalized CES case and the Cobb–Douglas case. The CES case allows for *any* direction of factor-augmenting technical change over the long run, and this direction is positively related to but not equal to the direction of R&D – if only the proposed direction of R&D can be sustained in the growth model (cf. Acemoglu, 2003). The Cobb–Douglas case implies that technical change will be always purely labor-augmenting over the long run (along the balanced growth path), regardless of the underlying direction of R&D.

The current paper has also developed a novel, tractable model of directed R&D, underlying the postulated unit factor productivity (UFP) distributions. Using this

model, we have provided a theoretical argument why the Weibull distribution should in fact be a good proxy of the real-world UFP distributions. The argument is based on the assumption that ideas (technologies, production techniques) are not simple, as it was implicitly assumed in earlier literature, but inherently complex, consisting of a large number of complementary components. Under such circumstances, total efficiency of a technology should be closely following the efficiency of its “weakest link”. The Weibull distribution is, in turn, an extreme value distribution pertaining to the minimum of a sequence of random draws from the same distribution which is bounded from below. Consequently, we have shown that if technologies consist of a wide range of complementary components, and they are then optimally chosen by firms, then the aggregate production function should be CES.

In the appendix, we also demonstrate that all our arguments are easily generalizable to the case of n -input production functions.

A few related issues may be studied in further research. It would be worthwhile, for example, to investigate the real-world productivity distributions, attempting to discriminate econometrically between the Weibull specification and the celebrated Pareto one (or perhaps some further distributions, too). Another challenge would be to develop an empirical approach able to identify jointly the parameters of the aggregate production function and the technology menu. It could also be interesting to see the consequences of allowing for dependence between the marginal Weibull distributions.

References

- [1] Acemoglu, D. (2003), “Labor- and Capital-Augmenting Technical Change,” *Journal of the European Economic Association* 1, 1-37.
- [2] Acemoglu, D. (2007), “Equilibrium Bias of Technology,” *Econometrica* 75(5), 1371-1409.
- [3] Acemoglu, D., V. Guerrieri (2008), “Capital Deepening and Non-Balanced Economic Growth,” *Journal of Political Economy* 116(3), 467-498.
- [4] Blanchard, O., M. Kremer (1997), “Disorganization,” *Quarterly Journal of Economics* 112(4), 1091-1126.
- [5] Caselli, F., W.J. Coleman (2006), “The World Technology Frontier,” *American Economic Review* 96(3), 499-522.
- [6] de Haan, L., A. Ferreira (2006), *Extreme Value Theory. An Introduction*. Springer.
- [7] Gabaix, X. (1999), “Zipf’s Law for Cities: An Explanation,” *Quarterly Journal of Economics* 114, 739-67.
- [8] Growiec, J. (2007), “Beyond the Linearity Critique: The Knife-Edge Assumption of Steady-State Growth,” *Economic Theory* 31(3), 489-499.
- [9] Growiec, J. (2008a), “A New Class of Production Functions and an Argument Against Purely Labor-Augmenting Technical Change”, *International Journal of Economic Theory* 4(4), 483-502.
- [10] Growiec, J. (2008b), “Production Functions and Distributions of Unit Factor Productivities: Uncovering the Link”, *Economics Letters* 101(1), 87-90.
- [11] Jones, C.I. (2005), “The Shape of Aggregate Production Functions and the Direction of Technical Change”, *Quarterly Journal of Economics* 120(2), 517-549.
- [12] Jones, C.I. (2011), “Intermediate Goods and Weak Links in the Theory of Economic Development”, *American Economic Journal: Macroeconomics* 3(2), 1-28.

- [13] Klump, R., P. McAdam, A. Willman (2011), “The Normalized CES Production Function: Theory and Empirics”, European Central Bank, WP No. 1294.
- [14] Klump, R., O. de La Grandville (2000), “Economic Growth and the Elasticity of Substitution: Two Theorems and Some Suggestions”, *American Economic Review* 90(1), 282-291.
- [15] Klump, R., H. Preissler (2000), “CES Production Functions and Economic Growth”, *Scandinavian Journal of Economics* 102(1), 41-56.
- [16] Kortum, S.S. (1997), “Research, Patenting, and Technological Change,” *Econometrica* 65, 1389-1419.
- [17] Kotz, S., S. Nadarajah (2000), *Extreme Value Distributions. Theory and Applications*. Imperial College Press, London.
- [18] Kremer, M. (1993), “The O-Ring Theory of Economic Development”, *Quarterly Journal of Economics* 108(3), 551-575.
- [19] La Grandville, O. de (1989), “In Quest of the Slutsky Diamond”, *American Economic Review* 79, 468-481.
- [20] Li, C.-W. (2000), “Endogenous vs. Semi-Endogenous Growth in a Two-R&D-Sector Model”, *Economic Journal* 110, C109–C122.
- [21] Uzawa, H. (1961), “Neutral Inventions and the Stability of Growth Equilibrium,” *Review of Economic Studies* 28(2), 117-124.

A Appendix. Generalization to n inputs and proofs of propositions

As announced in the main text, all the results provided here go through for n -input production functions as well. Let us now discuss this case.

A.1 The normalized CES case

First, let us show that if ideas (UFPs), augmenting each of the n production inputs, are *independently Weibull-distributed* (and the LPFs are normalized CES functions), then the resultant aggregate production is normalized CES as well. To this end, we shall use the following generalized assumptions. By x_i , $i = 1, 2, \dots, n$ we shall denote the inputs, and by a_i , $i = 1, 2, \dots, n$ – unit factor productivities.

Assumption 7 *The n -input local production function (LPF) takes either the normalized CES or the normalized Leontief form:*

$$Y = \begin{cases} Y_0 \left(\sum_{i=1}^n \pi_{0i} \left(\frac{a_i x_i}{a_{0i} x_{0i}} \right)^\theta \right)^{\frac{1}{\theta}}, & \text{if } \sigma_{LPF} \in (0, 1), \\ Y_0 \min_{i=1, \dots, n} \left\{ \left(\frac{a_i x_i}{a_{0i} x_{0i}} \right) \right\}, & \text{if } \sigma_{LPF} = 0, \end{cases} \quad (29)$$

where $\theta \in [-\infty, 0)$ is the substitutability parameter, related to the elasticity of substitution via $\sigma_{LPF} = \frac{1}{1-\theta}$. Leontief LPFs, with $\sigma_{LPF} = 0$, are obtained as a special case of the more general normalized CES class of LPFs by taking the limit $\theta \rightarrow -\infty$ (we denote this case as $\theta = -\infty$ for simplicity). π_{0i} is the income share of i -th factor at t_0 . Factor income shares sum up to unity:

$$\sum_{i=1}^n \pi_{0i} = 1, \quad (30)$$

and the LPF exhibits constant returns to scale.

Assumption 8 *The technology menu, specified in the (a_1, \dots, a_n) space, is given by the equality:*

$$H(a_1, \dots, a_n) = \sum_{i=1}^n \left(\frac{a_i}{\lambda_{ai}} \right)^\alpha = N, \quad \lambda_{a1}, \dots, \lambda_{an}, \alpha, N > 0. \quad (31)$$

The technology menu is understood as a contour line of the cumulative distribution function of the joint n -variate distribution of factor-augmenting ideas a_i , $i = 1, \dots, n$. Under independence of the n dimensions (so that marginal distributions are multiplied by one another), equation (31) obtains if and only if the marginal distributions are Weibull with the same shape parameter $\alpha > 0$ (Growiec, 2008b):²⁰

$$P(\tilde{a}_i > a_i) = e^{-\left(\frac{a_i}{\lambda_{ai}}\right)^\alpha}, \quad i = 1, 2, \dots, n,$$

where all $a_i > 0$. Under such parametrization, we have

$$P(\tilde{a}_1 > a_1, \dots, \tilde{a}_n > a_n) = e^{-\sum_{i=1}^n \left(\frac{a_i}{\lambda_{ai}}\right)^\alpha},$$

and thus the parameter N in equation (31) is interpreted as $N = -\ln P(\tilde{a}_1 > a_1, \dots, \tilde{a}_n > a_n) > 1$.

The case where $a_i, i = 1, \dots, n$ are independently Pareto distributed leads to a different specification of the technology menu and will be considered separately in the next subsection. If they are Weibull distributed but dependent, or independent but following some other distribution than Pareto or Weibull, the resultant aggregate production does not belong to the CES class and will not be considered here.

Assumption 9 *Firms choose the technology n -tuple (a_1, \dots, a_n) optimally, subject to the current technology menu, such that their profit is maximized:*

$$\max_{a,b} \left\{ Y_0 \left(\sum_{i=1}^n \pi_{0i} \left(\frac{a_i x_i}{a_{0i} x_{0i}} \right)^\theta \right)^{\frac{1}{\theta}} \right\} \quad s.t. \quad \sum_{i=1}^n \left(\frac{a_i}{\lambda_{ai}} \right)^\alpha = N. \quad (32)$$

Factor remuneration, taken into account in the firms' profit maximization problem, does not depend on the chosen technology so it can be safely omitted from the above optimization problem.²¹

Finally, second order conditions require us to assume that $\alpha > \theta$, so that the interior stationary point of the above problem is a maximum. For the resultant aggregate

²⁰Equation (2) can also be obtained for Pareto distributions of a_i , provided that the pattern of dependence between both marginal distributions is modeled with the Clayton copula (Growiec, 2008a). Its functional form has been first postulated by Caselli and Coleman (2006), but without any justification.

²¹In the case of Leontief LPFs, optimization requires $\frac{a_i x_i}{a_{0i} x_{0i}} = \frac{a_j x_j}{a_{0j} x_{0j}}$ for all $i, j = 1, \dots, n$.

production function to be concave with respect to x_i , $i = 1, \dots, n$, we need to assume furthermore that $\alpha - \theta - \alpha\theta > 0$. All these conditions are satisfied automatically in the case $\alpha > 0 > \theta$, on which we concentrate here. The inputs are gross complements in the aggregate production function.

Again, our framework provides direct results on the firm's optimal technology choice. First, at time t_0 , when $Y = Y_0$ and $x_i = x_{0i}$, $\lambda_{ai} = \lambda_{a0i}$ is assumed for all $i = 1, \dots, n$, the optimal choice satisfies:

$$a_{0i}^* = (N\pi_{0i})^{\frac{1}{\alpha}} \lambda_{a0i}, \quad i = 1, \dots, n, \quad (33)$$

where λ_{a0i} are the values of λ_{ai} at time t_0 . Values of a_{0i}^* will be used as a_{0i} in the normalization at the local level in all subsequent derivations.

For any other moment in time, the optimal technology choices are:

$$\left(\frac{a_j}{a_{0j}}\right)^* = \frac{\lambda_{aj}}{\lambda_{a0j}} \left(\sum_{i=1}^n \pi_{0i} \left(\frac{\lambda_{ai} \lambda_{a0j} x_i x_{0j}}{\lambda_{aj} \lambda_{a0i} x_{0i} x_j} \right)^{\frac{\alpha\theta}{\alpha-\theta}} \right)^{-\frac{1}{\alpha}}, \quad (34)$$

for all $j = 1, \dots, n$, where $\frac{\alpha\theta}{\alpha-\theta}$ is substituted with $-\alpha$ in the case of Leontief LPFs ($\theta = -\infty$).

Inserting these optimal technology choices into the LPF, we obtain the following result.

Proposition 4 *If Assumptions 7-9 hold, then the aggregate production function takes the normalized CES form:*

$$Y = Y_0 \left(\sum_{i=1}^n \pi_{0i} \left(\frac{\lambda_{ai} x_i}{\lambda_{a0i} x_{0i}} \right)^{\frac{\alpha\theta}{\alpha-\theta}} \right)^{\frac{\alpha-\theta}{\alpha\theta}}. \quad (35)$$

The normalized CES result obtains both in the case of CES and Leontief LPFs.

Proof. The proof is straightforward and requires just algebraic manipulations. To prove Proposition 1, one should simply take $n = 2$ in the following calculations.

First, in the case of CES LPFs, we form the Lagrangean:

$$\mathcal{L} = Y_0 \left(\sum_{i=1}^n \pi_{0i} \left(\frac{a_i x_i}{a_{0i} x_{0i}} \right)^\theta \right)^{\frac{1}{\theta}} + \Lambda \cdot \left\{ \sum_{i=1}^n \left(\frac{a_i}{\lambda_{ai}} \right)^\alpha - N \right\}. \quad (36)$$

Differentiating it with respect to a_i , $i = 1, \dots, n$, and getting rid of Λ yields:

$$\left(\frac{a_i}{a_j}\right)^{\alpha-\theta} = \frac{\pi_{0i}}{\pi_{0j}} \left(\frac{\lambda_{ai}}{\lambda_{aj}}\right)^\alpha \left(\frac{x_i a_{0j} x_{0j}}{x_j a_{0i} x_{0i}}\right)^\theta, \quad (37)$$

for all $i, j = 1, \dots, n$. Considering first the reference point of time t_0 , when $x_i = x_{0i}$, $\lambda_{ai} = \lambda_{a0i}$, $a_i = a_{0i}$ for all $i = 1, \dots, n$, we obtain:

$$\frac{a_{0i}}{a_{0j}} = \left(\frac{\pi_{0i}}{\pi_{0j}} \right)^{\frac{1}{\alpha}} \frac{\lambda_{a0i}}{\lambda_{a0j}}. \quad (38)$$

Using the specification of the technology menu (31) as well as the assumption that $\sum_{i=1}^n \pi_{0i} = 1$, we obtain:

$$a_{0i}^* = (N\pi_{0i})^{\frac{1}{\alpha}} \lambda_{a0i}, \quad i = 1, \dots, n. \quad (39)$$

For $t \neq t_0$, by plugging (39) into (37), using (31) again and rearranging, we obtain that:

$$\left(\frac{a_j}{a_{0j}} \right)^* = \frac{\lambda_{aj}}{\lambda_{a0j}} \left(\sum_{i=1}^n \pi_{0i} \left(\frac{\lambda_{ai}}{\lambda_{aj}} \frac{\lambda_{a0j}}{\lambda_{a0i}} \frac{x_i x_{0j}}{x_{0i} x_j} \right)^{\frac{\alpha\theta}{\alpha-\theta}} \right)^{-\frac{1}{\alpha}}, \quad (40)$$

for all $j = 1, \dots, n$.

Plugging this into the LPF (29) and rearranging, we obtain the final result.

Given our parametric assumptions, second-order conditions for the maximization of the Lagrangean hold. To demonstrate this, it is useful to note that maximizing \mathcal{L} is equivalent to minimizing the following transformed Lagrangean \mathcal{L}_{min} (where the maximand function is taken to the power $\theta < 0$ for simplicity):

$$\mathcal{L}_{min} = Y_0^\theta \sum_{i=1}^n \pi_{0i} \left(\frac{a_i x_i}{a_{0i} x_{0i}} \right)^\theta + \Lambda_{min} \cdot \left\{ \sum_{i=1}^n \left(\frac{a_i}{\lambda_{ai}} \right)^\alpha - N \right\}. \quad (41)$$

We obtain the following second-order derivatives (after inserting the first order condition to get rid of Λ_{min}):

$$\frac{\partial^2 \mathcal{L}_{min}}{\partial a_i^2} = \theta(\theta - \alpha) Y_0^\theta \pi_{0i} \left(\frac{a_i x_i}{a_{0i} x_{0i}} \right)^\theta \frac{1}{a_i^2} > 0, \quad (42)$$

$$\frac{\partial^2 \mathcal{L}_{min}}{\partial a_i \partial a_j} = 0, \quad (43)$$

and thus \mathcal{L}_{min} is minimized.

In the case of Leontief LPFs, instead of forming the Lagrangean, one should use the equality $\frac{a_i x_i}{a_{0i} x_{0i}} = \frac{a_j x_j}{a_{0j} x_{0j}}$ for all $i, j = 1, \dots, n$ – which must hold because of the

assumption that the representative firm maximizes profits. Since equations (31) and (39) still hold, plugging these equalities into the LPF yields

$$Y = Y_0 \frac{a_1 x_1}{a_{01} x_{01}} = Y_0 \left(\sum_{i=1}^n \pi_{0i} \left(\frac{\lambda_{ai} x_i}{\lambda_{a0i} x_{0i}} \right)^{-\alpha} \right)^{-\frac{1}{\alpha}}. \quad (44)$$

Please note that the same result is obtained by taking the case of CES LPFs and considering the limit $\theta \rightarrow -\infty$. ■

The corollary on factor income shares goes through in the n -dimensional case as well:

Corollary 3 *Assuming that factors are priced at their marginal product, the factor income shares are equal to:*

$$\pi_i = \frac{\pi_{0i} \left(\frac{\lambda_{ai} x_i}{\lambda_{a0i} x_{0i}} \right)^{\frac{\alpha\theta}{\alpha-\theta}}}{\sum_{i=1}^n \pi_{0i} \left(\frac{\lambda_{ai} x_i}{\lambda_{a0i} x_{0i}} \right)^{\frac{\alpha\theta}{\alpha-\theta}}}, \quad i = 1, \dots, n. \quad (45)$$

$$(46)$$

A.2 The Cobb–Douglas case

Let us now replace Assumption 8 with the following one:

Assumption 10 (modification of Assumption 8) *The technology menu, specified in the (a_1, \dots, a_n) space, is given by the equality:*

$$H(a_1, \dots, a_n) = \prod_{i=1}^n \left(\frac{a_i}{\lambda_{ai}} \right)^{\phi_i} = N, \quad \phi_i > 0, i = 1, \dots, n. \quad (47)$$

This shape of the technology menu is consistent with the assumption that a_i 's are independently Pareto-distributed, with shape parameters ϕ_i . In such case, $N = \frac{1}{P(\tilde{a}_1 > a_1, \dots, \tilde{a}_n > a_n)}$.

At t_0 , when $Y = Y_0$ and $x_i = x_{0i}$, $\lambda_{ai} = \lambda_{a0i}$ is assumed for $i = 1, \dots, n$, the optimal choice is indeterminate, provided that

$$\pi_{0i} = \frac{\phi_i}{\sum_{i=1}^n \phi_i}, \quad i = 1, \dots, n. \quad (48)$$

This restriction means that the factor income shares should be equal to $\frac{\phi_i}{\sum_{i=1}^n \phi_i}$. Thus, $\pi_{02}, \dots, \pi_{0n}$ cease to be free parameters, and a_{02}, \dots, a_{0n} become free parameters instead

(the remaining technology choice a_{01} is then calculated according to the technology menu).

At any other moment in time, and given $a_{0i}, i = 1, \dots, n$, the optimal technology choices are:

$$\left(\frac{a_i}{a_{0i}}\right)^* = \frac{x_{0i}}{x_i} \prod_{i=1}^n \left(\frac{\lambda_{ai} x_i}{\lambda_{a0i} x_{0i}}\right)^{\frac{\phi_i}{\sum_{i=1}^n \phi_i}}, \quad i = 1, \dots, n. \quad (49)$$

Inserting these optimal technology choices into the LPF, we obtain the following result.

Proposition 5 *If Assumptions 7, 9, and 10 hold, then the aggregate production function takes the Cobb–Douglas form:*

$$Y = Y_0 \prod_{i=1}^n \left(\frac{\lambda_{ai} x_i}{\lambda_{a0i} x_{0i}}\right)^{\frac{\phi_i}{\sum_{i=1}^n \phi_i}}. \quad (50)$$

Proof. The proof is, again, straightforward and requires just algebraic manipulations. To prove Proposition 2, one should take $n = 2$ in the following calculations.

First, we form the Lagrangean:

$$\mathcal{L} = Y_0 \left(\sum_{i=1}^n \pi_{0i} \left(\frac{a_i x_i}{a_{0i} x_{0i}} \right)^\theta \right)^{\frac{1}{\theta}} + \Lambda \cdot \left\{ \prod_{i=1}^n \left(\frac{a_i}{\lambda_{ai}} \right)^{\phi_i} - N \right\}. \quad (51)$$

Differentiating it with respect to $a_i, i = 1, \dots, n$, and getting rid of Λ yields:

$$\left(\frac{a_i x_i}{a_{0i} x_{0i}} \right)^\theta \frac{\pi_{0i} \phi_i}{\pi_{0j} \phi_j} = 1, \quad (52)$$

for all $i, j = 1, \dots, n$. Considering first the reference point of time t_0 , when $x_i = x_{0i}, \lambda_{ai} = \lambda_{a0i}, a_i = a_{0i}$ for all $i = 1, \dots, n$, we obtain:

$$\frac{\pi_{0i}}{\pi_{0j}} = \frac{\phi_i}{\phi_j}. \quad (53)$$

Using the assumption that $\sum_{i=1}^n \pi_{0i} = 1$, we obtain that at t_0 , optimal technology choice is indeterminate provided that:

$$\pi_{0i} = \frac{\phi_i}{\sum_{i=1}^n \phi_i}, \quad i = 1, \dots, n. \quad (54)$$

For $t \neq t_0$, by plugging (53) into (52), using (47) and rearranging, we obtain that:

$$\left(\frac{a_i}{a_{0i}}\right)^* = \frac{x_{0i}}{x_i} \prod_{i=1}^n \left(\frac{\lambda_{ai}}{\lambda_{a0i}} \frac{x_i}{x_{0i}}\right)^{\frac{\phi_i}{\sum_{i=1}^n \phi_i}}, \quad i = 1, \dots, n. \quad (55)$$

for all $j = 1, \dots, n$.

Plugging this into the LPF (29) and rearranging, we obtain the final result.

Given our parametric assumptions, second-order conditions for the maximization of the Lagrangean hold. To prove this, it is useful to note that maximizing \mathcal{L} is equivalent to minimizing the following transformed Lagrangean \mathcal{L}_{min} (where, for simplicity, the maximand function is taken to the power $\theta < 0$ and a log-transformation is applied to the restriction):

$$\mathcal{L}_{min} = Y_0^\theta \sum_{i=1}^n \pi_{0i} \left(\frac{a_i x_i}{a_{0i} x_{0i}}\right)^\theta + \Lambda_{min} \cdot \left\{ \sum_{i=1}^n \phi_i (\ln a_i - \ln \lambda_{ai}) - \ln N \right\}. \quad (56)$$

We obtain the following second-order derivatives (after inserting the first order condition to get rid of Λ_{min}):

$$\frac{\partial^2 \mathcal{L}_{min}}{\partial a_i^2} = \theta^2 Y_0^\theta \pi_{0i} \left(\frac{a_i x_i}{a_{0i} x_{0i}}\right)^\theta \frac{1}{a_i^2} > 0, \quad (57)$$

$$\frac{\partial^2 \mathcal{L}_{min}}{\partial a_i \partial a_j} = 0, \quad (58)$$

and thus \mathcal{L}_{min} is minimized.

In the case of Leontief LPFs, instead of forming the Lagrangean, one should use the equality $\frac{a_i x_i}{a_{0i} x_{0i}} = \frac{a_j x_j}{a_{0j} x_{0j}}$ for all $i, j = 1, \dots, n$ – which must hold because of the assumption that the representative firm maximizes profits. Since equation (47) still holds, plugging these equalities into the LPF yields

$$Y = Y_0 \frac{a_1 x_1}{a_{01} x_{01}} = Y_0 \prod_{i=1}^n \left(\frac{\lambda_{ai}}{\lambda_{a0i}} \frac{x_i}{x_{0i}}\right)^{\frac{\phi_i}{\sum_{i=1}^n \phi_i}}. \blacksquare \quad (59)$$