Narodowy Bank Polski

NBP Working Paper No. 201

Isoelastic elasticity of substitution production functions

Jakub Growiec

NBP Working Paper No. 201

Isoelastic elasticity of substitution production functions

Jakub Growiec

Economic Institute
Warsaw, 2015

Jakub Growiec – Narodowy Bank Polski and Warsaw School of Economics;
jakub.growiec@nbp.pl

The views expressed herein belong to the author and have not been endorsed by Narodowy Bank Polski.

Published by:
Narodowy Bank Polski
Education & Publishing Department
ul. Świętokrzyska 11/21
00-919 Warszawa, Poland
phone +48 22 185 23 35
www.nbp.pl

ISSN 2084-624X

© Copyright Narodowy Bank Polski, 2015

Contents

Abstract

We generalize the normalized Constant Elasticity of Substitution (CES) production function by allowing the elasticity of substitution to vary isoelastically with (i) relative factor shares, (ii) marginal rates of substitution, (iii) capital–labor ratios, or (iv) capital–output ratios. Ensuing four variants of Isoelastic Elasticity of Substitution (IEES) production functions have a range of intuitively desirable properties and yield empirically testable predictions for the functional relationship between relative factor shares and capital–labor ratios.

Keywords: production function, factor share, elasticity of substitution, marginal rate of substitution, normalization.

JEL Classification Numbers: E23, O47.

1 Introduction

The contemporary macroeconomic literature witnesses a revival of interest in the variation in the labor's share of GDP across countries and time. Understanding this variation calls for an analytical framework where factor (i.e., capital and labor) shares can be affected by endogenously determined variables. Probably the most popular among such frameworks is the one based on the Constant Elasticity of Substitution (CES) production function, first introduced to economics by Arrow, Chenery, Minhas, and Solow (1961). Identification of the true value of the elasticity of substitution σ between capital and labor is a notoriously difficult task, though. On the one hand, a voluminous literature exploiting time-series and cross-firm variation for the USA (Antràs, 2004; Chirinko, 2008; Klump, McAdam, and Willman, 2007, 2012; Young, 2013; Oberfield and Raval, 2014) finds that the elasticity of substitution is below unity ($\sigma \approx 0.6 - 0.7$), and thus both factors of production are gross complements. On the other hand, numerous studies exploiting the cross-country variation in factor shares (Duffy and Papageorgiou, 2000; Karabarbounis and Neiman, 2014; Piketty and Zucman, 2013; Piketty, 2014) tend to imply gross substitutability, with $\sigma \approx 1.2 - 1.3$.

Whether the factors are gross complements or substitutes is crucial both for long-run growth perspectives and short-run fluctuations, though. Above-unity elasticity of substitution can be perceived as an engine of long-run growth (Palivos and Karagiannis, 2010). If factors are gross substitutes then neither of them is essential for production, and thus physical capital accumulation alone can, under otherwise favorable circumstances, drive perpetual growth. Otherwise, the scarce factor limits the scope for economic development and output is bounded. Concurrently, the value of the elasticity of substitution has crucial implications for the immediate impact of factor accumulation on factor shares. Under gross substitutes, accumulation of capital relative to labor increases the capital's share of output. Under gross complements the opposite effect is observed. Hence, labor share declines observed across the world since the 1970-80s (Karabarbounis and Neiman, 2014) can be directly explained by capital deepening or capital-augmenting technological progress only if $\sigma > 1$.

It is questionable, however, if there is sufficient evidence to believe that the elasticity of substitution is a technological constant, unchanged by factor accumulation and factor-augmenting technological change. Indeed, relaxing this requirement could immediately resolve the apparent disagreement between σ estimates from studies based on US and cross-country factor share data. Imagine that, perhaps, the elasticity of substitution $\sigma(k)$ declines with the capital–labor ratio, being above unity in developing economies but below unity in developed ones such as the US. Although the discussed discrepancy in σ estimates might as well be due to various other reasons, this hypothesis seems to be a promising avenue for further inquiry. Moreover, it also has the advantage of consistency with the Kuznets curve (Kuznets, 1955; Galor, 2005): it implies that the capital share (and income inequality) should first increase (as long as $\sigma(k) > 1$), and then decrease (due to $\sigma(k) < 1$) across time in the course of a country's economic development. It also posits the caveat whether the current value of σ will persist forever, or – perhaps – could naturally fall below unity even if it were globally above unity now.

The current article puts forward and thoroughly characterizes a novel, tractable and empirically useful class of IsoElastic Elasticity of Substitution (IEES)[1] production functions, with constant returns to scale, which – under the restriction that $\psi < 0$ (to be discussed later) – can produce exactly the aforementioned result. As illustrated in Table 1, IEES functions generalize the CES function in the same way as the CES function generalizes the Cobb–Douglas. The Cobb–Douglas is isoelastic and implies constant factor shares, the CES function implies isoelastic factor shares and has a constant elasticity of substitution, whereas IEES functions have an isoelastic elasticity of substitution and a constant *elasticity of elasticity of substitution*. We then proceed to discuss their consistency with factor-augmenting technical change. Finally, we also provide an example illustrating their usefulness in empirical applications.

Table 1: How IEES Production Functions Generalize Cobb–Douglas and CES Functions

	$\sigma = 1$	$\frac{\sigma}{\sigma_0} = 1$	$\frac{\sigma}{\sigma_0} = \left(\frac{\pi}{1-\pi}\frac{1-\pi_0}{\pi_0}\right)^{\psi}$	$\frac{\sigma}{\sigma_0} = \left(\frac{MRS}{MRS_0}\right)^{\psi}$	$\frac{\sigma}{\sigma_0} = \left(\frac{k}{k_0}\right)^{\psi}$	$\frac{\sigma}{\sigma_0} = \left(\frac{k}{y}\frac{y_0}{k_0}\right)^{\psi}$
	C-D	**CES**	**IEES$\left(\frac{\pi}{1-\pi}\right)$**	**IEES(MRS)**	**IEES(k)**	**IEES(k/y)**
$\frac{\pi}{1-\pi}$	constant	C-D in k	CES in k	$\log \cdot k$	$\exp \cdot k$	$\pi \sim \exp \cdot \frac{k}{y}$
σ	1	constant	C-D in $\frac{\pi}{1-\pi}$	C-D in MRS	C-D in k	C-D in k/y

Notes: C–D denotes the Cobb–Douglas (isoelastic) function. $\frac{\pi}{1-\pi}$ is the relative factor share (the capital share π divided by the labor share $1-\pi$). σ is the elasticity of substitution. MRS is the marginal rate of substitution. Acronyms $\log \cdot k$, $\exp \cdot k$ and $\exp \cdot \frac{k}{y}$ will be explained later in text.

IEES production functions have a few notable advantages compared to functions with a variable elasticity of substitution (VES) which have already been analyzed in the literature. First, the class of IEES functions is sufficiently general to nest some

[1] Best pronounced as "yes". Abbreviation designed to avoid confusion with the intertemporal elasticity of substitution (IES).

of them directly, such as the Revankar's VES (1971) or the Stone–Geary production function (Geary, 1949-50; Stone, 1954). Second, as opposed to the empirically popular translog function (Christensen, Jorgenson, and Lau, 1973; Kim, 1992) or the empirically motivated VES function due to Lu (1967), it is not a local approximation of an arbitrary function but has well-behaved and economically interpretable properties globally. Third, alike the translog function but unlike VES production functions discussed in a wave of articles around 1970 (Lu, 1967; Sato and Hoffman, 1968; Kadiyala, 1972)[2] it naturally lends itself to further generalizations. For example, mirroring the extension from the Cobb–Douglas to the CES and from the CES to the IEES, the elasticity of elasticity of substitution could be made isoelastic instead of constant. One could thus eliminate one of the potential limitations of IEES functions: that $\sigma(k)$ is monotone in k.

All our calculations have been carried out in normalized units, thanks to which the role of each parameter of IEES functions has been precisely disentangled from all others, facilitating theoretical discussions as well as parameter estimation (Klump and de La Grandville, 2000; Klump, McAdam, and Willman, 2012).

The remainder of the article is structured as follows. Section 2 defines IEES production functions and derives their key properties. Sections 3–5 contain a detailed elaboration of three cases of IEES functions: where the elasticity of substitution is isoelastic with respect to the relative factor share, the marginal rate of substitution, and the factor ratio k. Section 6 complements the analysis with the *capital deepening* representation of the production function (Klenow and Rodriguez-Clare, 1997; Madsen, 2010) and elaborates on an IEES function where the elasticity of substitution is isoelastic with respect to the degree of capital deepening, k/y. Section 7 discusses the role of factor-augmenting technical change with IEES production. Section 8 illustrates the usefulness of IEES production functions in empirical applications. Section 9 concludes.

[2] See Mishra (2010) for a review of the history of production functions.

2 Definitions and Main Results

For any constant-returns-to-scale (CRS) production function F of two inputs, K and L, one can write $Y = F(K, L)$ in its intensive form $y = f(k)$, where $y = Y/L$ and $k = K/L$. We assume that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is three times continuously differentiable, increasing and concave in its whole domain.

All the analysis will be carried out in *normalized units*. While generally redundant for Cobb–Douglas production functions due to their multiplicative character, it has been shown for the case of CES functions (de La Grandville, 1989; Klump and de La Grandville, 2000) that production function normalization is crucial for obtaining clean identification of the role of each of its parameters. As we shall see shortly, the same argument applies equally forcefully to the proposed class of IEES functions.

The natural objects of comparison in the current study are the Cobb–Douglas and the CES production function. The normalized Cobb–Douglas function is written as:

$$y = f(k) = y_0 \left(\frac{k}{k_0}\right)^{\pi_0}, \qquad k_0, y_0 > 0, \pi_0 \in (0,1). \tag{1}$$

The normalized CES production function is, in turn:

$$y = f(k) = y_0 \left(\pi_0 \left(\frac{k}{k_0}\right)^{\frac{\sigma-1}{\sigma}} + (1-\pi_0)\right)^{\frac{\sigma}{\sigma-1}}, \qquad k_0, y_0 > 0, \pi_0 \in (0,1), \sigma > 0, \tag{2}$$

converging to the Cobb–Douglas function as the elasticity of substitution $\sigma \to 1$, to a linear function as $\sigma \to +\infty$, and to a Leontief (minimum) function as $\sigma \to 0_+$.

The following elementary concepts are central to our analysis:

- **Factor shares.** The partial elasticity of output Y with respect to K is defined as $\pi(k) = \frac{kf'(k)}{f(k)} \in [0,1]$. If markets are perfectly competitive, this elasticity is also equal to the capital's share of output. By constant returns to scale, implying that the labor share is $1 - \pi(k)$, it is also easily obtained that the ratio of factor shares (and of partial elasticities), strictly increasing in $\pi(k)$, is equal to $\frac{\pi(k)}{1-\pi(k)} = \frac{kf'(k)}{f(k)-kf'(k)} \geq 0$.

  The Cobb–Douglas production function is characterized by constant factor shares, with $\pi(k) \equiv \pi_0$ for all $k \geq 0$. For the CES production function, the ratio of factor shares $\frac{\pi(k)}{1-\pi(k)} = \frac{\pi_0}{1-\pi_0}\left(\frac{k}{k_0}\right)^{\frac{\sigma-1}{\sigma}}$ increases with k, from 0 when $k = 0$ to $+\infty$ as $k \to \infty$, if $\sigma > 1$. Conversely, if $\sigma < 1$ then the ratio gradually declines, from $+\infty$ towards 0.

- **Marginal rate of substitution (MRS).** For constant-returns-to-scale functions of two inputs, the MRS – capturing the slope of the isoquant – is computed as $MRS(k) \equiv \varphi(k) = -\frac{1-\pi(k)}{\pi(k)}k = -\frac{f(k)}{f'(k)} + k \leq 0$. Monotonicity and concavity of the production function f imply that the MRS is negative and (at least weakly) declines with k.

  The Cobb–Douglas function has a linearly declining MRS $\varphi(k) = \varphi_0 \left(\frac{k}{k_0}\right)$.[3] The CES function, in turn, has an isoelastic MRS $\varphi(k) = \varphi_0 \left(\frac{k}{k_0}\right)^{1/\sigma}$. Both functions unambiguously decline from 0 when $k = 0$ to $-\infty$ when $k \to \infty$.

- **Elasticity of substitution.** The elasticity of substitution – measuring the curvature of the isoquant, i.e., the elasticity of changes in the factor ratio k in reaction to changes in the MRS – is computed as $\sigma(k) = \frac{\varphi(k)}{k\varphi'(k)} = -\frac{f'(k)(f(k)-kf'(k))}{kf(k)f''(k)} \geq 0$. Concavity of the production function f implies that the elasticity of substitution is non-negative.

  The Cobb–Douglas function satisfies $\sigma(k) \equiv 1$ for all $k \geq 0$. For CES functions, the elasticity of substitution $\sigma > 0$ is a constant parameter.

The current study adopts the following definitions.

Definition 1 *The* elasticity of elasticity of substitution with respect to x, $EES(x)$, *is defined as the elasticity with which the elasticity of substitution σ reacts to changes in x:*

$$EES(x) = \frac{\partial\sigma(x)}{\partial x}\frac{x}{\sigma(x)} = \frac{\sigma'(k)}{\sigma(k)}\frac{x(k)}{x'(k)}, \tag{3}$$

where the last equality assumes that x is a differentiable function of k. We consider four arguments of the EES:

$$EES\left(\frac{\pi}{1-\pi}\right) = \frac{\pi(k)(1-\pi(k))}{\pi'(k)}\frac{\sigma'(k)}{\sigma(k)}, \tag{4}$$

$$EES(\varphi) = \frac{\varphi(k)}{\varphi'(k)}\frac{\sigma'(k)}{\sigma(k)} = k\sigma'(k), \tag{5}$$

$$EES(k) = \frac{k\sigma'(k)}{\sigma(k)}, \tag{6}$$

$$EES\left(\frac{k}{y}\right) = \frac{k}{1-\pi(k)}\frac{\sigma'(k)}{\sigma(k)}. \tag{7}$$

[3] Using the notation $\varphi_0 = -\frac{1-\pi_0}{\pi_0}k_0 < 0$.

Definition 2 *The* isoelastic elasticity of substitution production function $IEES(x)$ *is a function for which* $EES(x) \equiv const$.

In what follows, we shall characterize the four respective IEES functions, with $x \in \left\{\frac{\pi}{1-\pi}; \varphi; k; \frac{k}{y}\right\}$.[4] Please observe that for every CES or Cobb–Douglas function with a constant σ, $EES(x) = 0$ for all x, and thus they naturally belong to this wider class as well. Another observation is that EES is a *third-order characteristic* of any function f: existence of $\sigma'(k)$ for all k requires that f is at least three times differentiable in its domain. Standard axioms of production functions do not place any sign restrictions on $f^{(3)}(k)$ as well as on $EES(k)$, a degree of freedom that we shall exploit.

We are now in the position to spell out the main results of the current study.

Construction. The construction of a function f whose elasticity of substitution $\sigma(k)$ is of given form can be obtained in two steps: in the first step, $\sigma(k)$ is integrated up to yield the marginal rate of substitution $\varphi(k)$; in the second step $\varphi(k)$ is integrated up to yield the function $f(k)$ itself.[5] Formally,

$$\sigma(k) = \frac{\varphi(k)}{k\varphi'(k)} \quad \Rightarrow \quad \varphi(k) = -\exp\left(\int \frac{dk}{k\sigma(k)}\right), \tag{8}$$

$$\varphi(k) = -\frac{f(k)}{f'(k)} + k \quad \Rightarrow \quad f(k) = \exp\left(\int \frac{dk}{k - \varphi(k)}\right). \tag{9}$$

Both constants of integration have to be picked specifically to maintain production function normalization. For IEES production functions, integration (8) can be executed easily, yielding closed, economically interpretable formulas for the MRS as a function of k. In contrast, integration (9) generally cannot be performed in elementary functions – but for a few notable exceptions, some of which have already been discussed in the literature.

[4] The last case requires also a more general elaboration of the *capital deepening* production function representation, i.e. rewriting $y = f(k)$ in the form of $y = h(k/y)$.

[5] Solving it in a single step is also possible but requires solving a second-order nonlinear differential equation.

3 The IEES$\left(\frac{\pi}{1-\pi}\right)$ Function

The IEES$\left(\frac{\pi}{1-\pi}\right)$ production function, defined as a function for which $EES\left(\frac{\pi}{1-\pi}\right) = \psi$, where $\psi \in \mathbb{R}$ is a constant, implies (upon normalization) that the elasticity of substitution follows:

$$\frac{\sigma}{\sigma_0} = \left(\frac{\pi}{1-\pi}\frac{1-\pi_0}{\pi_0}\right)^{\psi}. \tag{10}$$

In this case, integration (8) yields the following formula for the MRS:

$$\varphi(k) = \varphi_0\left(\frac{1}{\sigma_0}\left(\frac{k}{k_0}\right)^{-\psi} + \left(1-\frac{1}{\sigma_0}\right)\right)^{-\frac{1}{\psi}}, \tag{11}$$

where $\varphi_0 = -\left(\frac{1-\pi_0}{\pi_0}\right)k_0$.

Hence, the relative factor share satisfies:

$$\frac{\pi}{1-\pi} = \frac{\pi_0}{1-\pi_0}\left(\frac{1}{\sigma_0} + \left(1-\frac{1}{\sigma_0}\right)\left(\frac{k}{k_0}\right)^{\psi}\right)^{\frac{1}{\psi}}. \tag{12}$$

Both above formulas demonstrate the symmetry, owing to which the IEES$\left(\frac{\pi}{1-\pi}\right)$ function is an equally natural generalization of the CES as the CES is a generalization of the Cobb–Douglas (isoelastic) production function. For the CES function, the MRS and relative factor shares are a Cobb–Douglas (isoelastic) function of k and the elasticity of substitution is constant. For the IEES$\left(\frac{\pi}{1-\pi}\right)$ function, the MRS and relative factor share are CES functions of k and the elasticity of substitution is Cobb–Douglas (isoelastic) in the relative factor share.

Inserting (12) back into (10) implies that the elasticity of substitution is the following function of k:

$$\sigma(k) = 1 + (\sigma_0 - 1)\left(\frac{k}{k_0}\right)^{\psi}, \tag{13}$$

and hence $\sigma(k) > 1$ for all k if $\sigma_0 > 1$, irrespective of the value of ψ, and conversely, $\sigma(k) < 1$ for all k if $\sigma_0 < 1$. Contrary to the hypothesis spelled out in the Introduction, capital and labor are either always gross substitutes or always gross complements here. Due to the strict monotonicity of the relative factor share with respect to k (equation (10)), the elasticity of substitution $\sigma(k)$ cannot cross unity. Moreover, the case $\sigma_0 = 1$ automatically reduces the IEES$\left(\frac{\pi}{1-\pi}\right)$ function directly to the Cobb–Douglas specification.

To further illustrate the properties of the current production function specification, we shall consider four specific cases, delineated by the assumptions made with respect to ψ and σ_0. We shall also discuss the special cases with $\psi = \pm 1$ for which integration (9) yields known closed-form formulas.[6] The case $\psi = 1$ corresponds to the "variable elasticity of substitution" (VES) production function due to Revankar (1971) whereas the case $\psi = -1$ captures the Stone–Geary production function.

Case with $\psi > 0$ and $\sigma_0 > 1$. In this case, factors of production are always gross substitutes and hence the capital share increases with the capital–labor ratio k. Since also the elasticity of substitution increases with the capital share, it follows that the elasticity of substitution increases with k as well. The production function is well-defined, increasing and concave in its entire domain $k \in [0, +\infty)$. We obtain the following limits:

$$\lim_{k\to 0} \frac{\pi(k)}{1-\pi(k)} = \frac{\pi_0}{1-\pi_0}\sigma_0^{-\frac{1}{\psi}} > 0, \qquad \lim_{k\to\infty} \frac{\pi(k)}{1-\pi(k)} = +\infty, \tag{14}$$

$$\lim_{k\to 0} \varphi(k) = 0, \qquad \lim_{k\to\infty} \varphi(k) = \varphi_0 \left(\frac{\sigma_0}{\sigma_0 - 1}\right)^{\frac{1}{\psi}} < 0, \tag{15}$$

$$\lim_{k\to 0} \sigma(k) = 1, \qquad \lim_{k\to\infty} \sigma(k) = +\infty. \tag{16}$$

Case with $\psi < 0$ and $\sigma_0 > 1$. In this case, factors of production are always gross substitutes and hence the capital share increases with the capital–labor ratio k. Since the elasticity of substitution, on the other hand, decreases with the capital share, it follows that the elasticity of substitution decreases with k as well. The production function is well-defined, increasing and concave in its entire domain $k \in [0, +\infty)$. We obtain the following limits:

$$\lim_{k\to 0} \frac{\pi(k)}{1-\pi(k)} = 0, \qquad \lim_{k\to\infty} \frac{\pi(k)}{1-\pi(k)} = \frac{\pi_0}{1-\pi_0}\sigma_0^{-\frac{1}{\psi}} > 0, \tag{17}$$

$$\lim_{k\to 0} \varphi(k) = \varphi_0 \left(\frac{\sigma_0}{\sigma_0 - 1}\right)^{\frac{1}{\psi}} < 0, \qquad \lim_{k\to\infty} \varphi(k) = -\infty, \tag{18}$$

$$\lim_{k\to 0} \sigma(k) = +\infty, \qquad \lim_{k\to\infty} \sigma(k) = 1. \tag{19}$$

Case with $\psi > 0$ and $\sigma_0 < 1$. In this case, factors of production are always gross complements and hence the capital share is inversely related to the capital–labor ratio

[6] Symbolic integration reveals that closed-form formulas (albeit huge and generally difficult to interpret) exist also for $\psi = \pm 2, \pm\frac{1}{2}, -3$. They are available from the author upon request.

k. Since the elasticity of substitution, on the other hand, increases with the capital share, it follows that the elasticity of substitution falls with k. The production function is well-defined, increasing and concave only for $k \in [0, k_{max}]$, where $k_{max} = k_0(1 - \sigma_0)^{-1/\psi}$. We obtain the following limits:

$$\lim_{k\to 0} \frac{\pi(k)}{1-\pi(k)} = \frac{\pi_0}{1-\pi_0}\sigma_0^{-\frac{1}{\psi}} > 0, \qquad \lim_{k\to k_{max}} \frac{\pi(k)}{1-\pi(k)} = 0, \tag{20}$$

$$\lim_{k\to 0} \varphi(k) = 0, \qquad \lim_{k\to k_{max}} \varphi(k) = -\infty, \tag{21}$$

$$\lim_{k\to 0} \sigma(k) = 1, \qquad \lim_{k\to k_{max}} \sigma(k) = 0. \tag{22}$$

Case with $\psi < 0$ and $\sigma_0 < 1$. In this case, factors of production are always gross complements and hence the capital share is inversely related to the capital–labor ratio k. Since also the elasticity of substitution is inversely related to the capital share, it follows that the elasticity of substitution increases with k. The production function is well-defined, increasing and concave only for $k \in [k_{min}, +\infty)$, where $k_{min} = k_0(1 - \sigma_0)^{-1/\psi}$. We obtain the following limits:

$$\lim_{k\to k_{min}} \frac{\pi(k)}{1-\pi(k)} = +\infty, \qquad \lim_{k\to\infty} \frac{\pi(k)}{1-\pi(k)} = \frac{\pi_0}{1-\pi_0}\sigma_0^{-\frac{1}{\psi}} > 0, \tag{23}$$

$$\lim_{k\to k_{min}} \varphi(k) = 0, \qquad \lim_{k\to\infty} \varphi(k) = -\infty, \tag{24}$$

$$\lim_{k\to k_{min}} \sigma(k) = 0, \qquad \lim_{k\to\infty} \sigma(k) = 1. \tag{25}$$

Revankar's VES production function. Assuming that $\psi = 1$, following Revankar (1971), allows us to find the antiderivative in (9) in elementary functions. The normalized "variable elasticity of substitution" (Revankar's VES) production function with constant returns to scale reads:

$$y = f(k) = y_0 \left(\frac{k}{k_0}\right)^{\frac{\pi_0}{\pi_0+\sigma_0(1-\pi_0)}} \left(\pi_0 \left(\frac{\sigma_0 - 1}{\sigma_0}\right)\left(\frac{k}{k_0}\right) + \frac{\pi_0 + \sigma_0(1-\pi_0)}{\sigma_0}\right)^{\frac{\sigma_0(1-\pi_0)}{\pi_0+\sigma_0(1-\pi_0)}}, \tag{26}$$

or in non-normalized notation, $f(k) = Ak^{\alpha}(Bk+1)^{1-\alpha}$, with $\alpha \in (0,1)$, $A > 0$ and $B \in \mathbb{R}$. Please observe the domain restriction $k \leq -1/B$ which is in force if $B < 0$ (i.e., $\sigma_0 < 1$).

It is notable that while several of the production functions derived around 1970, which do not belong to the class of IEES functions, have remained something of a theoretical curiosity, the Revankar's VES function has been repeatedly used in empirical studies, even quite recently (Karagiannis, Palivos, and Papageorgiou, 2005).

Stone–Geary production function. Assuming that $\psi = -1$ also allows us to find the antiderivative in (9) in elementary functions. The normalized Stone–Geary production function (i.e., Cobb–Douglas production function of a shifted input) is:

$$y = f(k) = y_0 \left(\left(\frac{k}{k_0} \right) \left(\frac{\sigma_0 \pi_0 + (1 - \pi_0)}{\sigma_0} \right) + (1 - \pi_0) \frac{\sigma_0 - 1}{\sigma_0} \right)^{\frac{\sigma_0 \pi_0}{\sigma_0 \pi_0 + (1 - \pi_0)}}, \qquad (27)$$

or in non-normalized notation, $f(k) = A(k + B)^{\alpha}$, with $\alpha \in (0, 1)$, $A > 0$ and $B \in \mathbb{R}$. Please observe the domain restriction $k \geq -B$ which is in force if $B < 0$ (i.e., $\sigma_0 < 1$).

4 The IEES(MRS) Function

The IEES(MRS) production function, defined as a function for which $EES(\varphi) = \psi$, where $\psi \in \mathbb{R}$ is a constant, implies (upon normalization) that the elasticity of substitution follows:

$$\frac{\sigma}{\sigma_0} = \left(\frac{\varphi}{\varphi_0}\right)^{\psi}, \tag{28}$$

where $\varphi_0 = -\left(\frac{1-\pi_0}{\pi_0}\right) k_0$.

In this case, integration (8) yields the following formula for the MRS:

$$\varphi(k) = \varphi_0 \left(1 + \frac{\psi}{\sigma_0} \ln\left(\frac{k}{k_0}\right)\right)^{\frac{1}{\psi}}. \tag{29}$$

Hence, the relative factor share satisfies:

$$\frac{\pi}{1-\pi} = \frac{\pi_0}{1-\pi_0} \frac{k}{k_0} \left(1 + \frac{\psi}{\sigma_0} \ln\left(\frac{k}{k_0}\right)\right)^{-\frac{1}{\psi}} \tag{30}$$

Inspection of the above formulas reveals that the MRS is a logarithmic function of k. The relative factor share is, on the other hand, a product of a logarithmic and a linear function of k. As opposed to the cases of the Cobb–Douglas, CES, and IEES$\left(\frac{\pi}{1-\pi}\right)$ functions, relative factor shares are no longer a monotonic function of k. There exists a unique point of reversal, coinciding with the point where the elasticity of substitution crosses unity, $\tilde{k} = k_0 e^{-\frac{\sigma_0 - 1}{\psi}}$ with $\sigma(\tilde{k}) = 1$. The hypothesis spelled out in the Introduction requires that $\psi < 0$ and thus at $\tilde{k}$, capital and labor switch from being gross substitutes to gross complements.

Inserting (30) back into (28) implies that the elasticity of substitution is the following function of k:

$$\sigma(k) = \sigma_0 + \psi \ln\left(\frac{k}{k_0}\right). \tag{31}$$

To further illustrate the properties of the current production function specification, we shall consider two specific cases, delineated by the assumptions made with respect to ψ. Unfortunately, to our knowledge, IEES(MRS) functions cannot be obtained in a closed form.

Case with $\psi > 0$. In this case, the elasticity of substitution decreases with the marginal rate of substitution ($\varphi_0 < 0$) and thus increases with the factor ratio k (recall that by concavity and constant returns to scale, the MRS necessarily decreases with k). The production function is well-defined, increasing and concave only for $k \in [k_{min}, +\infty)$, where $k_{min} = k_0 e^{-\sigma_0/\psi}$. The relative factor share $\frac{\pi(k)}{1-\pi(k)}$ (and thus the capital's share $\pi(k)$ as well) follows a non-monotonic pattern with k, declining if $k \in (k_{min}, \tilde{k})$ and increasing for $k > \tilde{k}$. The minimum capital share, obtained at the point $\tilde{k}$, is equal to:

$$\left(\frac{\pi}{1-\pi}\right)_{min} = \frac{\pi(\tilde{k})}{1-\pi(\tilde{k})} = \frac{\pi_0}{1-\pi_0} e^{-\frac{\sigma_0 - 1}{\psi}} \sigma_0^{\frac{1}{\psi}}. \tag{32}$$

We also obtain the following limits:

$$\lim_{k \to k_{min}} \frac{\pi(k)}{1-\pi(k)} = +\infty, \qquad \lim_{k \to \infty} \frac{\pi(k)}{1-\pi(k)} = +\infty, \tag{33}$$

$$\lim_{k \to k_{min}} \varphi(k) = 0, \qquad \lim_{k \to \infty} \varphi(k) = -\infty, \tag{34}$$

$$\lim_{k \to k_{min}} \sigma(k) = 0, \qquad \lim_{k \to \infty} \sigma(k) = +\infty. \tag{35}$$

Case with $\psi < 0$. In this case, the elasticity of substitution increases with the marginal rate of substitution and thus falls with the factor ratio k. The production function is well-defined, increasing and concave only for $k \in [0, k_{max}]$, where $k_{max} = k_0 e^{-\sigma_0/\psi}$. The relative factor share $\frac{\pi(k)}{1-\pi(k)}$ (and thus the capital's share $\pi(k)$ as well) follows a non-monotonic pattern with k, increasing when $k \in (0, \tilde{k})$ and falling for $k \in (\tilde{k}, k_{max})$. The maximum capital share, obtained at the point $\tilde{k}$, is equal to:

$$\left(\frac{\pi}{1-\pi}\right)_{max} = \frac{\pi(\tilde{k})}{1-\pi(\tilde{k})} = \frac{\pi_0}{1-\pi_0} e^{-\frac{\sigma_0 - 1}{\psi}} \sigma_0^{\frac{1}{\psi}}. \tag{36}$$

We also obtain the following limits:

$$\lim_{k \to 0} \frac{\pi(k)}{1-\pi(k)} = 0, \qquad \lim_{k \to k_{max}} \frac{\pi(k)}{1-\pi(k)} = 0, \tag{37}$$

$$\lim_{k \to 0} \varphi(k) = 0, \qquad \lim_{k \to k_{max}} \varphi(k) = -\infty, \tag{38}$$

$$\lim_{k \to 0} \sigma(k) = +\infty, \qquad \lim_{k \to k_{max}} \sigma(k) = 0. \tag{39}$$

5 The IEES(k) Function

The IEES(k) production function, defined as a function for which $EES(k) = \psi$, where $\psi \in \mathbb{R}$ is a constant, implies (upon normalization) that the elasticity of substitution follows:

$$\frac{\sigma}{\sigma_0} = \left(\frac{k}{k_0}\right)^{\psi}. \tag{40}$$

In this case, integration (8) yields the following formula for the MRS:

$$\varphi(k) = \varphi_0 e^{\frac{1}{\psi\sigma_0}\left(1-\left(\frac{k}{k_0}\right)^{-\psi}\right)}, \tag{41}$$

where $\varphi_0 = -\left(\frac{1-\pi_0}{\pi_0}\right) k_0$.

Hence, the relative factor share satisfies:

$$\frac{\pi}{1-\pi} = \frac{\pi_0}{1-\pi_0}\frac{k}{k_0}e^{-\frac{1}{\psi\sigma_0}\left(1-\left(\frac{k}{k_0}\right)^{-\psi}\right)} \tag{42}$$

Inspection of the above formulas reveals that the MRS is an exponential function of k. Relative factor shares are, on the other hand, a product of an exponential and a linear function of k. As opposed to the cases of the Cobb–Douglas, CES, and IEES$\left(\frac{\pi}{1-\pi}\right)$ functions, and alike the IEES(MRS) function, the relative factor share is a non-monotonic function of k. There is a unique point of reversal, coinciding with the point where the elasticity of substitution crosses unity, $\tilde{k} = k_0\sigma_0^{-1/\psi}$ with $\sigma(\tilde{k}) = 1$. The hypothesis spelled out in the Introduction requires that $\psi < 0$ and thus at $\tilde{k}$, capital and labor switch from being gross substitutes to gross complements.

To further illustrate the properties of the current production function specification, we shall consider two specific cases, delineated by the assumptions made with respect to ψ. Unfortunately, to our knowledge, IEES(k) functions cannot be obtained in a closed form.

Case with $\psi > 0$. In this case, we assume that the elasticity of substitution increases with the factor ratio k. The production function is well-defined, increasing and concave in its domain $k \in [0, +\infty)$. The relative factor share $\frac{\pi(k)}{1-\pi(k)}$ (and thus the capital's share $\pi(k)$ as well) follows a non-monotonic pattern with k, declining if $k \in (0, \tilde{k})$ and increasing for $k > \tilde{k}$. The minimum capital share, obtained at the point $\tilde{k}$, is equal to:

$$\left(\frac{\pi}{1-\pi}\right)_{min} = \frac{\pi(\tilde{k})}{1-\pi(\tilde{k})} = \frac{\pi_0}{1-\pi_0}e^{-\frac{\sigma_0-1}{\psi\sigma_0}}\sigma_0^{-\frac{1}{\psi}}. \tag{43}$$

We also obtain the following limits:

$$\lim_{k\to 0} \frac{\pi(k)}{1-\pi(k)} = +\infty, \qquad \lim_{k\to\infty} \frac{\pi(k)}{1-\pi(k)} = +\infty, \tag{44}$$

$$\lim_{k\to 0} \varphi(k) = 0, \qquad \lim_{k\to\infty} \varphi(k) = \varphi_0 e^{\frac{1}{\psi\sigma_0}} < 0, \tag{45}$$

$$\lim_{k\to 0} \sigma(k) = 0, \qquad \lim_{k\to\infty} \sigma(k) = +\infty. \tag{46}$$

Case with $\psi < 0$. In this case, we assume that the elasticity of substitution decreases with the factor ratio k. The production function is well-defined, increasing and concave in its domain $k \in [0, +\infty)$. The relative factor share $\frac{\pi(k)}{1-\pi(k)}$ (and thus the capital's share $\pi(k)$ as well) follows a non-monotonic pattern with k, increasing when $k \in (0, \tilde{k})$ and falling for $k > \tilde{k}$. The maximum capital share, obtained at the point $\tilde{k}$, is equal to:

$$\left(\frac{\pi}{1-\pi}\right)_{max} = \frac{\pi(\tilde{k})}{1-\pi(\tilde{k})} = \frac{\pi_0}{1-\pi_0} e^{-\frac{\sigma_0 - 1}{\psi\sigma_0}} \sigma_0^{-\frac{1}{\psi}}. \tag{47}$$

We also obtain the following limits:

$$\lim_{k\to 0} \frac{\pi(k)}{1-\pi(k)} = 0, \qquad \lim_{k\to\infty} \frac{\pi(k)}{1-\pi(k)} = 0, \tag{48}$$

$$\lim_{k\to 0} \varphi(k) = \varphi_0 e^{\frac{1}{\psi\sigma_0}} < 0, \qquad \lim_{k\to\infty} \varphi(k) = -\infty, \tag{49}$$

$$\lim_{k\to 0} \sigma(k) = +\infty, \qquad \lim_{k\to\infty} \sigma(k) = 0. \tag{50}$$

6 The Capital Deepening Production Function Representation and the IEES(k/y) Function

It is popular, especially in the growth and development accounting literature (see e.g., Klenow and Rodriguez-Clare, 1997; Madsen, 2010), to rewrite the aggregate production function so that it takes $\kappa \equiv K/Y = k/y$ instead of k as its input. Increases in κ are then identified with *capital deepening*. The key reason for making such a transformation is that, unlike k, the capital deepening term κ should not exhibit a strong upward trend, and dealing with variables without discernible trends has its documented statistical advantages. And indeed, relative stability of the capital–output ratio (one of the "great ratios") has been long taken as a stylized fact, together with relative stability of factor shares. Only relatively recently have both postulates been questioned; still, if y and k exhibit upward trends, by definition k/y *must* be at least growing much slower than k, underscoring the empirical value of the current representation.

As a preliminary remark, observe how easy it is to rewrite the normalized Cobb–Douglas and CES functions in the capital deepening form:

$$y = y_0 \left(\frac{\kappa}{\kappa_0}\right)^{\frac{\pi_0}{1-\pi_0}}, \qquad \kappa_0, y_0 > 0, \pi_0 \in (0,1), \tag{51}$$

$$y = y_0 \left(\frac{1}{1-\pi_0} - \frac{\pi_0}{1-\pi_0}\left(\frac{\kappa}{\kappa_0}\right)^{\frac{\sigma-1}{\sigma}}\right)^{-\frac{\sigma}{\sigma-1}}, \ \kappa_0, y_0 > 0, \pi_0 \in (0,1), \sigma > 0. \tag{52}$$

The implied relative factor shares $\Pi(\kappa) = \frac{\pi(\kappa)}{1-\pi(\kappa)}$ are, respectively, equal to $\Pi(\kappa) = \frac{\pi_0}{1-\pi_0}$ (a constant) in the Cobb–Douglas case, and

$$\Pi(\kappa) = \frac{\frac{\pi_0}{1-\pi_0}\left(\frac{\kappa}{\kappa_0}\right)^{\frac{\sigma-1}{\sigma}}}{\frac{1}{1-\pi_0} - \frac{\pi_0}{1-\pi_0}\left(\frac{\kappa}{\kappa_0}\right)^{\frac{\sigma-1}{\sigma}}}, \qquad \pi(\kappa) = \pi_0\left(\frac{\kappa}{\kappa_0}\right)^{\frac{\sigma-1}{\sigma}} \tag{53}$$

in the CES case. Hence, the capital share increases with the degree of capital deepening κ if and only if $\sigma > 1$, i.e., if capital and labor are gross substitutes. Finally, observe that the functional form of equation (53) does not by itself preclude cases with $\pi(\kappa) > 1$. These cases are made impossible only by the range of the CES function which restricts the support of $\kappa = k/y$ appropriately.

Although rewriting the function and its implied elasticities in terms of κ for arbitrary (increasing and concave) production functions is not so easy anymore, it can always be done. Let us now recall some known relevant results.

Existence. Any increasing, concave, and constant-returns-to-scale (CRS) production function of two inputs, $Y = F(K, L)$, can be rewritten as $F\left(\frac{K}{Y}, \frac{L}{Y}\right) = 1$. Then, by the implicit function theorem,[7] there exists a function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\frac{L}{Y} = \frac{1}{h(K/Y)}$ and thus $y = h(\kappa)$. Note that due to concavity of F, the capital deepening term κ is always increasing in k. We also observe that the relative factor share can be computed directly as the elasticity of $h(\kappa)$ with respect to κ:

$$\Pi(\kappa) = \frac{\pi(\kappa)}{1-\pi(\kappa)} = \frac{h'(\kappa)\kappa}{h(\kappa)}. \tag{54}$$

The existence of an *explicit* form of the function $h(\kappa)$, however, hinges on the requirement that $F(\kappa, 1/y) = 1$ can be solved for y explicitly, which need not be the case even if the functional form of F is given. Notably, it *cannot* be done for IEES functions whose explicit form is not known.[8]

Construction. Using this notation, the proposed two-step method for finding functions whose elasticity of substitution is given as a predefined function of the degree of capital deepening κ is as follows:

$$\sigma(\kappa) = \frac{1}{1 - \frac{\Pi'(\kappa)\kappa}{\Pi(\kappa)(1+\Pi(\kappa))}} \quad \Rightarrow \quad \Pi(\kappa) = \frac{1}{\exp\left(-\int \frac{\sigma(\kappa)-1}{\kappa\sigma(\kappa)} d\kappa\right) - 1}, \tag{55}$$

$$\Pi(\kappa) = \frac{h'(\kappa)\kappa}{h(\kappa)} \quad \Rightarrow \quad h(\kappa) = \exp\left(\int \frac{\Pi(\kappa)}{\kappa} d\kappa\right). \tag{56}$$

Unfortunately, the integrals (55)–(56) can be computed in elementary functions only for a very narrow set of functional specifications of $\sigma(\kappa)$.

Still, this apparatus enables us to define and characterize the IEES(κ) production function whose elasticity of substitution is isoelastic in the degree of capital deepening.

The IEES(κ) production function, defined as a function for which $EES(\kappa) = \psi$, where $\psi \in \mathbb{R}$ is a constant, implies (upon normalization) that the elasticity of substitution follows:

$$\frac{\sigma}{\sigma_0} = \left(\frac{\kappa}{\kappa_0}\right)^{\psi} = \left(\frac{k}{y}\frac{y_0}{k_0}\right)^{\psi}. \tag{57}$$

[7] Which can be used because F is increasing and concave in its entire domain.

[8] It can be done for the special cases of Revankar's VES and Stone-Geary production function, though. Details are available upon request.

In this case, integration (55) yields the following formula for the relative factor share:

$$\Pi(\kappa) = \frac{\pi_0 \left(\frac{\kappa}{\kappa_0}\right)}{e^{\frac{1}{\psi\sigma_0}\left(1-\left(\frac{\kappa}{\kappa_0}\right)^{-\psi}\right)} - \pi_0 \left(\frac{\kappa}{\kappa_0}\right)}. \tag{58}$$

Slight rearrangement of the above formula reveals that the capital share $\pi(\kappa)$ is a product of an exponential and a linear function of κ:

$$\pi(\kappa) = \pi_0 \left(\frac{\kappa}{\kappa_0}\right) e^{-\frac{1}{\psi\sigma_0}\left(1-\left(\frac{\kappa}{\kappa_0}\right)^{-\psi}\right)}. \tag{59}$$

As opposed to the cases of the Cobb–Douglas, CES, and IEES$\left(\frac{\pi}{1-\pi}\right)$ functions, and alike the IEES(MRS) and IEES(k) functions, relative factor shares are a non-monotonic function of κ here (and thus, owing to the concavity of $F(K, L)$, of the capital–labor ratio k as well). There exists a unique point of reversal, coinciding with the point where the elasticity of substitution crosses unity, $\tilde{\kappa} = \kappa_0 \sigma_0^{-1/\psi}$ with $\sigma(\tilde{\kappa}) = 1$. The hypothesis spelled out in the Introduction requires that $\psi < 0$ and thus at $\tilde{\kappa}$, capital and labor switch from being gross substitutes to gross complements.

To further illustrate the properties of the current production function specification, we shall consider two specific cases, delineated by the assumptions made with respect to ψ. Unfortunately, to our knowledge, IEES(κ) functions cannot be obtained in a closed form.

Case with $\psi > 0$. In this case, we assume that the elasticity of substitution increases with the degree of capital deepening κ. Due to restrictions in the range of $F(K, L)$, the support of κ is restricted to $\kappa \in [\kappa_{min}, \kappa_{max}]$ where κ_{min} and κ_{max} are the two solutions to the equation $\pi(\kappa) = 1$. The capital's share $\pi(\kappa)$ follows a non-monotonic pattern with κ, declining if $\kappa \in (\kappa_{min}, \tilde{\kappa})$ and increasing for $\kappa \in (\tilde{\kappa}, \kappa_{max})$. The minimum capital share, obtained at the point $\tilde{\kappa}$, is equal to:

$$\pi_{min} = \pi(\tilde{\kappa}) = \pi_0 e^{-\frac{\sigma_0 - 1}{\psi\sigma_0}} \sigma_0^{-\frac{1}{\psi}}. \tag{60}$$

Case with $\psi < 0$. In this case, we assume that the elasticity of substitution decreases with the degree of capital deepening κ. The capital's share $\pi(k)$ follows a non-monotonic pattern with κ, increasing when $\kappa \in (0, \tilde{\kappa})$ and falling for $\kappa > \tilde{\kappa}$. The maximum capital share, obtained at the point $\tilde{\kappa}$, is equal to:

$$\pi_{max} = \pi(\tilde{\kappa}) = \pi_0 e^{-\frac{\sigma_0 - 1}{\psi\sigma_0}} \sigma_0^{-\frac{1}{\psi}}, \tag{61}$$

with the following limits:

$$\lim_{\kappa \to 0} \pi(\kappa) = 0, \qquad \lim_{\kappa \to \infty} \pi(\kappa) = 0. \tag{62}$$

The production function is well-defined, increasing and concave in its domain $\kappa \in [0, +\infty)$ as long as $\pi_{max} \leq 1$.

7 Factor-Augmenting Technical Change

One of the key advantages of assuming constant returns to scale lies with a clean treatment of factor-augmenting technical change. With just a slight modification of notation, technical change can be incorporated in CRS production functions by replacing $Y = F(K, L)$ with $Y = F(bK, aL)$, or – in the intensive form – by replacing $y = f(k)$ with $\bar{y} = f(\bar{k})$, where $\bar{y} = \frac{Y}{aL}$ and $\bar{k} = \frac{bK}{aL}$. Crucially, owing to constant returns to scale, the functional form of f remains unchanged. And if one is ultimately interested in y instead of $\bar{y}$, then one may simply compute $y = a\bar{y} = af(\bar{k}) = F(bk, a)$ after all the necessary derivations.

This last step implicitly separates the Hicks-neutral component of technical change from the *capital bias* in technical change (cf., e.g., León-Ledesma, McAdam, and Willman, 2010). This is the key insight for the current study because it allows us to define the capital share $\pi(\bar{k})$, the marginal rate of substitution $\varphi(\bar{k})$ and, crucially, the elasticity of substitution $\sigma(\bar{k})$, as a function of the capital–labor ratio *in effective units*. Hence, any capital-biased technical change (i.e., increase in b/a) acts just like physical capital accumulation, whereas labor-biased technical change (decline in b/a) affects factor shares, MRS and σ alike a decline in the capital–labor ratio k.[9] All functional forms remain unchanged.

Factor-augmenting technical change can be studied in the capital deepening production function representation as well. With the notation $\bar{\kappa} = \frac{\bar{k}}{\bar{y}} = \frac{bk}{y}$, one can easily replace $y = h(\kappa)$ with $\bar{y} = h(\bar{\kappa})$ and all the above results still go through. At the same time, this specification emphasizes that capital-augmenting technical change adds to capital deepening just like capital accumulation, while labor-augmenting technical change is neutral for capital deepening.

Clearly, both theory and data suggest that labor-augmenting technical change are likely to be dominant over the long run (Acemoglu, 2003; Klump, McAdam, and Willman, 2012), and therefore in effective units, the capital–labor ratio $\bar{k}$ will likely grow slower (if at all) than the raw capital–labor ratio k. Hence, for empirical applications of IEES functions (and CES ones as well), it is important whether one considers the capital–labor ratio in effective units ($\bar{k}$) or just as a raw variable, measured in dollars per worker (k).

[9]See Growiec (2013) for a discussion of the micro-level forces behind the direction of factor-augmenting technical change.

8 Usefulness in Empirical Applications

Usefulness of the proposed class of IEES production functions in empirical applications follows from the fact that they provide testable predictions for the functional relationships between two observables: factor shares and the capital-labor ratio k (or the degree of capital deepening, κ). Each of the nonlinear equations (12), (30), (42) and (59) can be estimated directly based on country-level, sectoral-level, or firm-level data.

Moreover, an important advantage of production function normalization which we use here is that each of the aforementioned four specifications can be used to determine simultaneously the magnitude of ψ and the *average* elasticity of substitution in the sample, σ_0. As for the latter parameter, the CES specification works as a natural benchmark for comparisons.

It must be kept in mind, however, that when the explicit functional form of the estimated IEES function is not known, empirical exercises cannot use numerous sophisticated estimation procedures developed in the associated econometric literature, such as the Kmenta (1967) second-order CES production function approximation around a predefined normalization point (see also Klump, McAdam, and Willman, 2012), or joint estimation of a normalized supply-side CES system with generalized Box-Cox technical progress terms (Klump, McAdam, and Willman, 2007).

On the other hand, the advantage of using factor share equations (i.e., (12), (30), (42) and (59)) instead of the production function specification directly is that such a transformation allows us to represent the CES function in a linear form whose slope is pinned down by the elasticity of substitution:

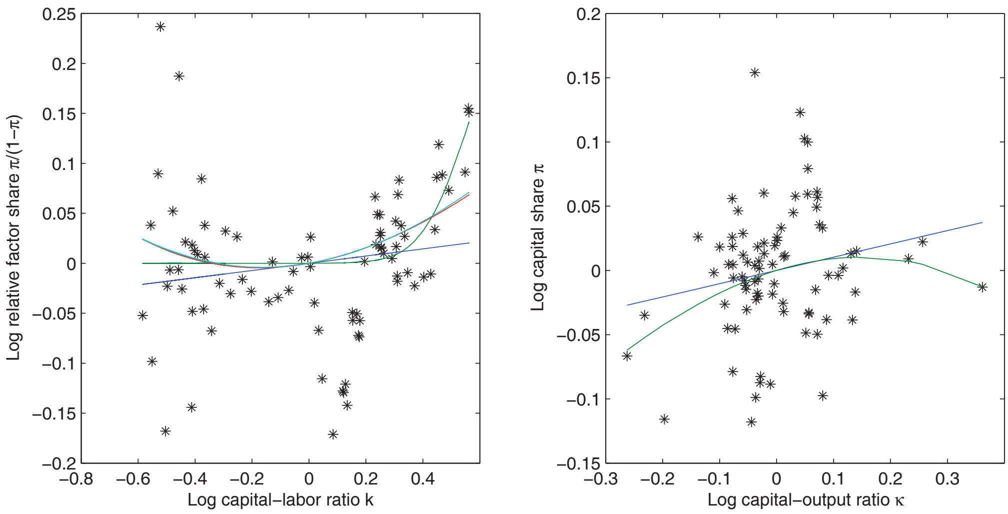
$$\ln\left(\frac{\pi}{1-\pi}\right) = \ln\left(\frac{\pi_0}{1-\pi_0}\right) + \frac{\sigma-1}{\sigma}\ln\left(\frac{k}{k_0}\right). \tag{63}$$

Hence, any nonlinearity detected between the log relative factor share and the log factor ratio in the course of IEES function estimation should be interpreted as an indication that the elasticity of substitution is in fact not constant in the data.

Figures 1–3 contain three examples of such empirical applications, for (i) long US time-series data (1929–2011), (ii) a wide cross-country dataset for 1996, and (iii) an annual panel of 17 Eurozone economies in 1996–2014. The data on factor shares, the capital–labor ratio k and output have been taken from Growiec, McAdam, and Mućk (2015), Caselli (2005) and the Ameco database[10], respectively. For each of these exam-

[10] http://ec.europa.eu/economy_finance/ameco/user/serie/SelectSerie.cfm [accessed 22.12.2014]

Figure 1: Nonlinear Least Squares Estimates of IEES Production Functions. Annual Time-Series Data: USA (1929–2011).



Notes. Left panel: blue line – CES; green line – IEES($\frac{\pi}{1-\pi}$); red line – IEES(MRS); tourquoise line – IEES(k). Right panel: blue line – CES; green line – IEES(κ). Data source: Growiec, McAdam, and Mućk (2015).

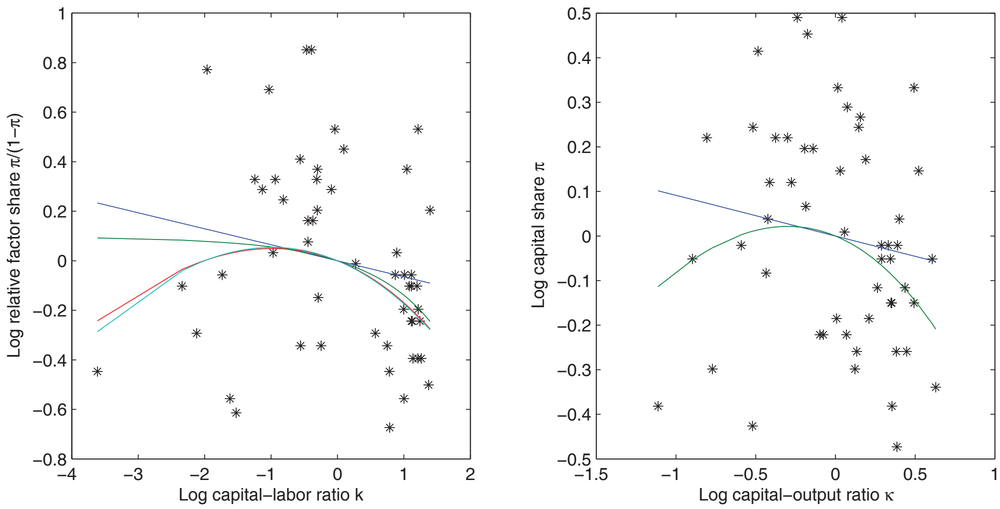
ples, we have normalized the data on the stock of capital per worker k, capital share of output π and the capital–output ratio κ by dividing the observations by respective sample-wide geometric averages. We have then estimated ψ and σ_0 from equations (12), (30), and (42), based on data on $\frac{\pi}{1-\pi}\frac{1-\pi_0}{\pi_0}$ and $\frac{k}{k_0}$, as well as (59) based on data on $\frac{\pi}{\pi_0}$ and $\frac{\kappa}{\kappa_0}$. For comparison, we have also included CES-based estimates of the elasticity of substitution. To improve estimation efficiency, given the multiplicative character of the stochastic disturbance term, we have applied nonlinear least squares to log-transformed versions of all equations.

The key feature of our results is the remarkable consistency of average elasticity of substitution (σ_0) estimates across various production function specifications. We also find the IEES(MRS) and IEES(k) functions to yield quantitatively very similar outcomes. Our other results are as follows (Table 2):

- **US Time-Series Data (1929–2011).** We find that the average elasticity of substitution σ_0 is slightly above and not statistically significantly different from unity.[11] The elasticity of σ with respect to k is moderately positive, and with re-

[11] This estimate is likely upward biased due to the neglect of capital-augmenting technical change (Klump, McAdam, and Willman, 2012).

Figure 2: Nonlinear Least Squares Estimates of IEES Production Functions. Cross-Country Data (1996).



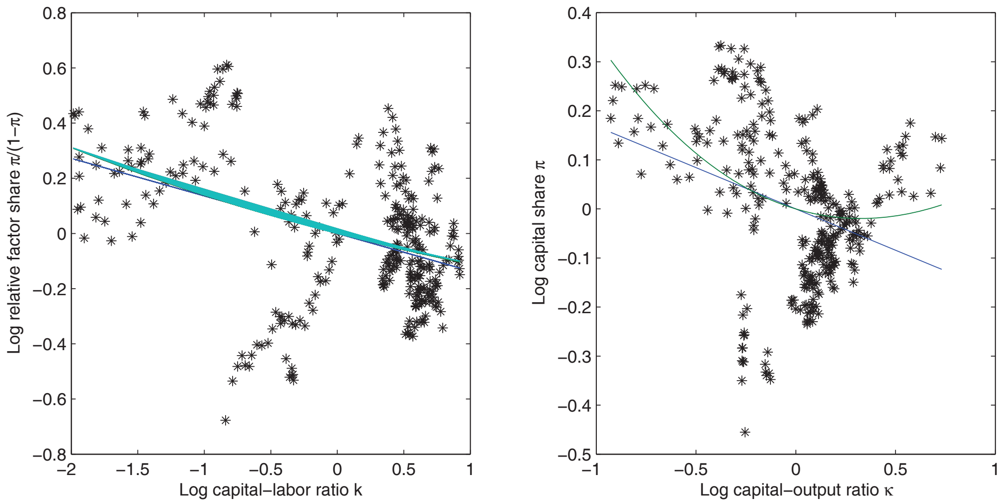
Notes. Left panel: blue line – CES; green line – IEES($\frac{\pi}{1-\pi}$); red line – IEES(MRS); tourquoise line – IEES(k). Right panel: blue line – CES; green line – IEES(κ). Data source: Caselli (2005).

spect to the degree of capital deepening κ – moderately negative. The IEES$\left(\frac{\pi}{1-\pi}\right)$ function appears to fit the data best (based on the Bayesian Information Criterion and R^2) but this result is only an artifact of the unreasonably high ψ estimate, fitting the upward swing in the capital share observed in the last few years of the sample. IEES(κ) outperforms the CES specification in terms of the BIC.

- **Cross-Sectional Data (1996).** σ_0 is below unity ($\sigma_0 \approx 0.9$) and either insignificant or marginally significant (depending on the specification). The elasticity of σ with respect to k and κ is negative but very modest. The IEES(k) specification is the best in terms of BIC and R^2. IEES(κ) improves upon the CES with respect to the BIC only slightly.

- **Panel Data, 17 Eurozone Countries (1996–2014).** We find $\sigma_0 \approx 0.89$, significantly below unity. All specifications provide very similar outcomes and ψ does not significantly differ from zero. Hence the CES specification outperforms three IEES variants in terms of the BIC. The elasticity of σ with respect to κ is, on the other hand, is significantly positive in this sample, and IEES(κ) outperforms the CES in terms of the BIC.

Figure 3: Nonlinear Least Squares Estimates of IEES Production Functions. Panel Data: Euro Zone (1996–2014).



Notes. Left panel: blue line – CES; green line – IEES$\left(\frac{\pi}{1-\pi}\right)$; red line – IEES(MRS); tourquoise line – IEES(k). Right panel: blue line – CES; green line – IEES(κ). Data source: Ameco database.

Our estimates can also provide predictions for the variation in the elasticity of substitution σ across time and space. For instance, the IEES(k) specification implies that σ exhibits a generally increasing time trend in the US, rising from the low of 0.88 in 1943, crossing unity in 1958 and reaching 1.25 in 2011. In the 1996 cross-country sample, the same specification implies that σ varies from 0.77 in Norway and Switzerland to 1.33 in Burundi and is very close to unity in Colombia and Paraguay. Inconsistently with the time-series dataset, it also implies $\sigma \approx 0.8$ in the USA in 1996.

Needless to say, econometric estimation of σ_0 and ψ in the above exercise can be improved in various ways. As shown by León-Ledesma, McAdam, and Willman (2010) in a CES framework, omitting factor-augmenting technical change (with a non-zero capital-augmenting part) leads the estimates of σ based on time-series variation to be biased towards unity. This is likely the case for our above findings, especially the ones based on long US data. Allowing for capital-augmenting technical change could help alleviate this bias.

Moreover, for estimates based on cross-country variation in data, a multitude of other factors beside the capital–labor ratio could also affect observed factor shares. For example, taxes, labor market institutions, sectoral structure of the economy, or skill composition can vary largely across countries. In panel data, the fixed compo-

Table 2: Summary of NLS Estimates of IEES Production Functions.

	CES(k)	IEES$\left(\frac{\pi}{1-\pi}\right)$	IEES(MRS)	IEES(k)	CES(κ)	IEES(κ)
			US TIME-SERIES DATA, 1929–2011			
σ_0	1.037	1.001	1.054	1.052	1.115	1.156
	(0.025)	(0.005)	(0.027)**	(0.026)*	(0.070)	(0.081)*
ψ		15.511	0.310	0.306		-0.968
		(11.673)	(0.116)***	(0.111)***		(0.560)*
R^2	0.028	0.181	0.114	0.116	0.039	0.071
BIC	-195.08	-204.90	-198.34	-196.59	-261.92	-260.33
			CROSS-SECTIONAL DATA, 1996			
σ_0	0.939	0.923	0.905	0.899	0.916	0.871
	(0.040)	(0.055)	(0.037)**	(0.038)**	(0.069)	(0.063)**
ψ		0.837	-0.098	-0.108		-0.461
		(1.003)	(0.045)**	(0.046)**		(0.241)*
R^2	0.038	0.093	0.135	0.139	0.023	0.098
BIC	51.14	52.02	49.50	49.23	4.43	4.16
			PANEL DATA, 17 EUROZONE COUNTRIES, 1996-2014			
σ_0	0.880	0.890	0.890	0.890	0.856	0.880
	(0.013)***	(0.017)***	(0.016)***	(0.016)***	(0.020)***	(0.022)***
ψ		-0.193	0.025	0.029		0.251
		(0.204)	(0.023)	(0.027)		(0.084)***
R^2	0.178	0.181	0.181	0.181	0.091	0.118
BIC	-21.69	-16.96	-17.04	-17.05	-318.50	-322.41

Notes: standard errors in parentheses. *** p<0.01, ** p<0.05, * p<0.1.
For σ_0, the null hypothesis of the t-test is $\sigma_0 = 1$.

nent of these differences can be controlled for when estimating the elasticity of substitution in first differences (Karabarbounis and Neiman, 2014). Unfortunately, unlike the CES case, for IEES functions log relative factor shares are not linear in the log capital–labor ratio. Hence, the level of this ratio is indispensable in estimation even if one adopts the first–differences specification.

Finally, biases in estimates of σ_0 and ψ could also obtain from mismeasurement of factor shares and capital endowments. Both these variables are constructed based on a range of assumptions (e.g., constant markups, the same labor share among the self-employed, constant capital depreciation rates, constant utilization rates across the business cycle) whose likely violation can also lead to systematic errors (Gollin, 2002; Mućk, McAdam, and Growiec, 2015; Fernald, 2014).

However, the objective of the current section was only to illustrate the usefulness of IEES functions in empirical studies. Addressing the above caveats systematically is left for further research.

9 Conclusion

In the current paper, we have constructed a novel class of normalized Isoelastic Elasticity of Substitution (IEES) production functions and analyzed its properties. Our analytical results are summarized in the following Table 3, expanding upon Table 1 provided in the Introduction. We have also discussed the empirical usefulness of these functions and the ways in which they can be reconciled with factor-augmenting technical change. Some of our preliminary empirical results are in accordance with the hypothesis spelled out in the Introduction – that in the course of ongoing capital accumulation, the elasticity of substitution declines and there exists a point where capital and labor switch from being gross substitutes to gross complements.

Further applications of IEES functions can be, at least, twofold. First, from the empirical perspective, they could improve our understanding of the dynamic behavior of factor shares over time as well as their dispersion across countries, regions, sectors, and firms. They could also turn out useful in growth and levels accounting.

Second, from the point of view of theory, they may become a useful tool for analyzing long-run growth, medium-run swings, and short-run fluctuations in economic activity. In particular, allowing for an endogenous shift between capital and labor being gross complements and gross substitutes (as it is possible for IEES(MRS), IEES(k) and IEES(κ) functions) can substantially change long-run predictions of some known growth models.

Table 3: Summary of Results

	$\sigma = 1$ **C-D**	$\frac{\sigma}{\sigma_0} = 1$ **CES**	$\frac{\sigma}{\sigma_0} = \left(\frac{\pi}{1-\pi}\frac{1-\pi_0}{\pi_0}\right)^{\psi}$ **IEES**$\left(\frac{\pi}{1-\pi}\right)$	$\frac{\sigma}{\sigma_0} = \left(\frac{\varphi(k)}{\varphi_0}\right)^{\psi}$ **IEES(MRS)**	$\frac{\sigma}{\sigma_0} = \left(\frac{k}{k_0}\right)^{\psi}$ **IEES(k)**
			MAIN RESULTS		
Domain	$k \in [0, +\infty)$	$k \in [0, +\infty)$	$\sigma_0 > 1: \ k \in [0, +\infty)$ $\sigma_0 < 1, \psi > 0: \ k \in [0, k_{max}]$ $\sigma_0 < 1, \psi < 0: \ k \in [k_{min}, +\infty)$	$\psi > 0: \ k \in [k_{min}, +\infty)$ $\psi < 0: \ k \in [0, k_{max}]$	$k \in [0, +\infty)$
$\frac{\pi(k)}{1-\pi(k)}$ (relative factor share)	$\frac{\pi_0}{1-\pi_0}$ constant	$\frac{\pi_0}{1-\pi_0}\left(\frac{k}{k_0}\right)^{\frac{\sigma-1}{\sigma}}$ $\sigma > 1$: increasing $\sigma < 1$: decreasing	$\frac{\pi_0}{1-\pi_0}\left(\frac{1}{\sigma_0} + \left(1 - \frac{1}{\sigma_0}\right)\left(\frac{k}{k_0}\right)^{\psi}\right)^{\frac{1}{\psi}}$ $\sigma_0 > 1$: increasing $\sigma_0 < 1$: decreasing	$\frac{\pi_0}{1-\pi_0}\frac{k}{k_0}\left(1 + \frac{\psi}{\sigma_0}\ln\left(\frac{k}{k_0}\right)\right)^{-\frac{1}{\psi}}$ $\psi > 0$: $\cup$-shaped $\psi < 0$: $\cap$-shaped	$\frac{\pi_0}{1-\pi_0}\frac{k}{k_0}e^{-\frac{1}{\psi\sigma_0}\left(1-\left(\frac{k}{k_0}\right)^{-\psi}\right)}$ $\psi > 0$: $\cup$-shaped $\psi < 0$: $\cap$-shaped
MRS, $\varphi(k)$	$\varphi_0\left(\frac{k}{k_0}\right)$	$\varphi_0\left(\frac{k}{k_0}\right)^{\frac{1}{\sigma}}$	$\varphi_0\left(\frac{1}{\sigma_0}\left(\frac{k}{k_0}\right)^{-\psi} + \left(1 - \frac{1}{\sigma_0}\right)\right)^{-\frac{1}{\psi}}$	$\varphi_0\left(1 + \frac{\psi}{\sigma_0}\ln\left(\frac{k}{k_0}\right)\right)^{\frac{1}{\psi}}$	$\varphi_0 e^{\frac{1}{\psi\sigma_0}\left(1-\left(\frac{k}{k_0}\right)^{-\psi}\right)}$
$\sigma(k)$ (elasticity of substitution)	1 constant	σ_0 constant	$1 + (\sigma_0 - 1)\left(\frac{k}{k_0}\right)^{\psi}$ $\sigma < 1(> 1) \Leftrightarrow \sigma_0 < 1(> 1)$ $\psi(\sigma_0 - 1) > 0$: increasing $\psi(\sigma_0 - 1) < 0$: decreasing	$\sigma_0 + \psi\ln\left(\frac{k}{k_0}\right)$ $\sigma = 1$ if $\frac{k}{k_0} = e^{-\frac{\sigma_0-1}{\psi}}$ $\psi > 0$: increasing $\psi < 0$: decreasing	$\sigma_0\left(\frac{k}{k_0}\right)^{\psi}$ $\sigma = 1$ if $\frac{k}{k_0} = \sigma_0^{-\frac{1}{\psi}}$ $\psi > 0$: increasing $\psi < 0$: decreasing
			RESULTS FOR THE CAPITAL DEEPENING REPRESENTATION (WITH $\kappa = k/y$)		
	C-D	**CES**	**IEES**(κ)		
$\pi(\kappa)$ (capital share)	π_0 constant	$\pi_0\left(\frac{\kappa}{\kappa_0}\right)^{\frac{\sigma-1}{\sigma}}$ $\sigma > 1$: increasing $\sigma < 1$: decreasing	$\pi_0\left(\frac{\kappa}{\kappa_0}\right)e^{-\frac{1}{\psi\sigma_0}\left(1-\left(\frac{\kappa}{\kappa_0}\right)^{-\psi}\right)}$ $\psi > 0$: $\cup$-shaped $\psi < 0$: $\cap$-shaped		
$\sigma(\kappa)$ (elasticity of substitution)	1 constant	σ_0 constant	$\sigma_0\left(\frac{\kappa}{\kappa_0}\right)^{\psi}$ $\sigma = 1$ if $\frac{\kappa}{\kappa_0} = \sigma_0^{-\frac{1}{\psi}}$ $\psi > 0$: increasing $\psi < 0$: decreasing		

References

ACEMOGLU, D. (2003): "Labor-and Capital-Augmenting Technical Change," *Journal of the European Economic Association*, 1, 1–37.

ANTRÀS, P. (2004): "Is the U.S. Aggregate Production Function Cobb–Douglas? New Estimates of the Elasticity of Substitution," *Contributions to Macroeconomics*, 4.

ARROW, K., H. B. CHENERY, B. S. MINHAS, AND R. M. SOLOW (1961): "Capital-Labor Substitution and Economic Efficiency," *Review of Economics and Statistics*, 43, 225–250.

CASELLI, F. (2005): "Accounting for Cross-Country Income Differences," in *Handbook of Economic Growth*, ed. by P. Aghion, and S. Durlauf. North-Holland.

CHIRINKO, R. S. (2008): "σ: The Long and Short of It," *Journal of Macroeconomics*, 30, 671–686.

CHRISTENSEN, L. R., D. W. JORGENSON, AND L. J. LAU (1973): "Transcendental Logarithmic Production Frontiers," *Review of Economics and Statistics*, 55, 29–45.

DE LA GRANDVILLE, O. (1989): "In Quest of the Slutsky Diamond," *American Economic Review*, 79, 468–481.

DUFFY, J., AND C. PAPAGEORGIOU (2000): "The Specification of the Aggregate Production Function: A Cross–Country Empirical Investigation," *Journal of Economic Growth*, 5, 83–116.

FERNALD, J. (2014): "Productivity and Potential Output Before, During, and After the Great Recession," in *NBER Macroeconomics Annual 2014, Volume 29*, NBER Chapters. National Bureau of Economic Research, Inc.

GALOR, O. (2005): "From Stagnation to Growth: Unified Growth Theory," in *Handbook of Economic Growth*, ed. by P. Aghion, and S. Durlauf, vol. 1 of *Handbook of Economic Growth*, chap. 4, pp. 171–293. Elsevier.

GEARY, R. C. (1949-50): "A Note on 'A Constant Utility Index of the Cost of Living'," *Review of Economic Studies*, 18, 65–66.

GOLLIN, D. (2002): "Getting Income Shares Right," *Journal of Political Economy*, 110, 458–474.

GROWIEC, J. (2013): "A Microfoundation for Normalized CES Production Functions with Factor-Augmenting Technical Change," *Journal of Economic Dynamics and Control*, 37, 2336–2350.

GROWIEC, J., P. MCADAM, AND J. MUĆK (2015): "Endogenous Labor Share Cycles: Theory and Evidence," Working Paper No. 1765, European Central Bank.

KADIYALA, K. R. (1972): "Production Functions and Elasticity of Substitution," *Southern Economic Journal*, 38, 281–284.

KARABARBOUNIS, L., AND B. NEIMAN (2014): "The Global Decline of the Labor Share," *The Quarterly Journal of Economics*, 129(1), 61–103.

KARAGIANNIS, G., T. PALIVOS, AND C. PAPAGEORGIOU (2005): "A Variable Elasticity of Substitution (VES) Aggregate Production Function: Theory and Evidence," in *New Trends in Macroeconomics*, ed. by C. Diebolt, and C. Kyrtsou. Springer Verlag, New York, NY.

KIM, H. Y. (1992): "The Translog Production Function and Variable Returns to Scale," *Review of Economics and Statistics*, 74, 546–552.

KLENOW, P. J., AND A. RODRIGUEZ-CLARE (1997): "The Neoclassical Revival in Growth Economics: Has It Gone Too Far?," in *NBER Macroeconomics Annual 1997*, ed. by B. Bernanke, and J. Rotemberg. MIT Press.

KLUMP, R., AND O. DE LA GRANDVILLE (2000): "Economic Growth and the Elasticity of Substitution: Two Theorems and Some Suggestions," *American Economic Review*, 90, 282–291.

KLUMP, R., P. MCADAM, AND A. WILLMAN (2007): "Factor Substitution and Factor Augmenting Technical Progress in the US: A Normalized Supply-Side System Approach," *Review of Economics and Statistics*, 89, 183–192.

——— (2012): "Normalization in CES Production Functions: Theory and Empirics," *Journal of Economic Surveys*, 26, 769–799.

KMENTA, J. (1967): "On Estimation of the CES Production Function," *International Economic Review*, 8, 180–189.

KUZNETS, S. (1955): "Economic Growth and Income Inequality," *American Economic Review*, 43, 1–28.

León-Ledesma, M. A., P. McAdam, and A. Willman (2010): "Identifying the Elasticity of Substitution with Biased Technical Change," *American Economic Review*, 100, 1330–1357.

Lu, Y.-C. (1967): *Variable Elasticity of Substitution Production Functions, Technical Change and Factor Shares*. Iowa State University, Ph.D. Thesis.

Madsen, J. B. (2010): "The Anatomy of Growth in the OECD Since 1870," *Journal of Monetary Economics*, 57(6), 753–767.

Mishra, S. K. (2010): "A Brief History of Production Functions," *IUP Journal of Managerial Economics*, 4, 6–34.

Mućk, J., P. McAdam, and J. Growiec (2015): "Understanding the Dynamics of the US Labor Share," Working Paper (forthcoming), European Central Bank.

Oberfield, E., and D. Raval (2014): "Micro Data and Macro Technology," Working Paper 20452, National Bureau of Economic Research.

Palivos, T., and G. Karagiannis (2010): "The Elasticity Of Substitution As An Engine Of Growth," *Macroeconomic Dynamics*, 14(05), 617–628.

Piketty, T. (2014): *Capital in the Twenty-First Century*. Harvard University Press, Translated from French by Arthur Goldhammer.

Piketty, T., and G. Zucman (2013): "Capital is Back: Wealth-Income Ratios in Rich Countries, 1700-2010," CEPR Discussion Papers 9588, C.E.P.R. Discussion Papers.

Revankar, N. S. (1971): "A Class of Variable Elasticity of Substitution Production Functions," *Econometrica*, 39, 61–71.

Sato, R., and R. F. Hoffman (1968): "Production Functions with Variable Elasticity of Factor Substitution: Some Analysis and Testing," *Review of Economics and Statistics*, 50, 453–460.

Stone, J. R. N. (1954): "Linear Expenditure Systems and Demand Analysis: An Application to the Pattern of British Demand," *Economic Journal*, pp. 511–527.

Young, A. T. (2013): "U.S. Elasticities Of Substitution And Factor Augmentation At The Industry Level," *Macroeconomic Dynamics*, 17(04), 861–897.

www.nbp.pl