

NBP Working Paper No. 311

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## Abstract

Labor's share of income has attracted interest in recent years reflecting its apparent protracted decline. These falls, witnessed across many countries, are usually deemed undesirable. Any such assertion, however, begs the question of what is the socially optimal labor share. We address this question using a micro-founded endogenous growth model calibrated on US data. We find that in our central calibration the socially optimal labor share is 17% (11 pp) above the decentralized equilibrium, calibrated to match the average observed in history. We also study the dependence of both long-run growth equilibria on model parameters and relate our results to Piketty's "laws of capitalism". Finally, we demonstrate that cyclical movements in factor income shares are socially optimal and that the decentralized equilibrium typically does not generate excess volatility.

**Keywords:** Labor income share, Endogenous growth, Factor augmenting endogenous technical change, Social optimum, Decentralized allocation.

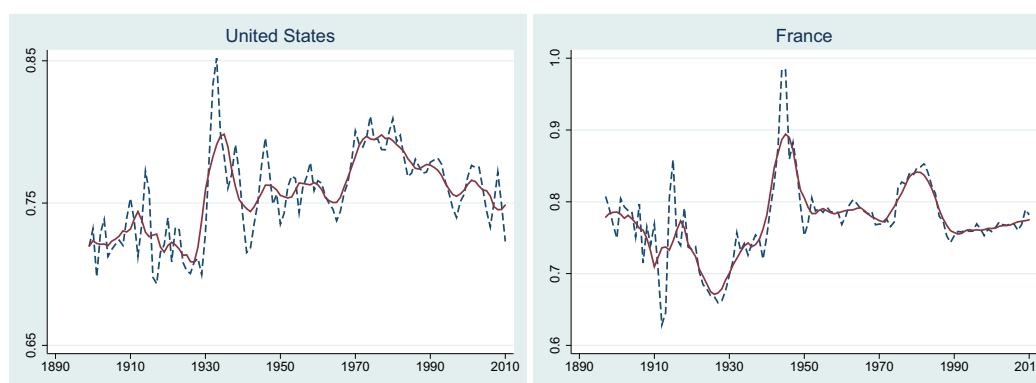
**JEL Codes:** O33, O41.

## 1 Introduction

Although interest in labor's share of income has a long tradition in economics, current interest has crystallized around its apparent fall in recent decades. Such falls, witnessed across many countries, are often presented in normative terms: namely as undesirable, potentially requiring intervention. Any such assertion, however, begs the question of what is the appropriate point of comparison. Can we say, for instance, if there is a socially optimal labor share?

Moreover, what of its dynamics? We know from historical sources such as Piketty and Zucman (2014) that labor income shares – even over extremely long horizons (e.g., > 100 years) – can fluctuate considerably (see **Figure 1** for the US and France). Both aspects matter for any normative discussion on the labor share. For instance if the labor share is falling, yet still above its 'optimal' level (or fluctuating around it) then, arguably, this might be interpreted passively, as a manifestation of recognized fluctuations in factor shares (e.g., Mućk et al. (2018)). Indeed, given that long and persistent fluctuations in the labor share are observed in practice, we might also wonder whether such fluctuations are socially optimal. By comparison, can we say if the decentralized labor share is characterized by excessive volatility?<sup>1</sup>

**FIGURE 1: HISTORICAL LABOR SHARE: US (1899-2010) & FRANCE (1897-2010)**



**Notes:** The French data is taken from Piketty and Zucman (2014). The US data also taken from Piketty and Zucman (2014) over the sample 1929-2010; prior to 1929 the labor share is extrapolated using the database by Groth and Madsen (2016) which provide compensation of employees and value added data starting in 1898 based on historical source provided by Liesner (1989). The **blue dashed line** is the level of the labor income share and the **red line** is a simple moving average process approximating its trend characteristics:  $\frac{1}{10} \sum_{j=-5}^4 l_{s_{t-j}}$ , where  $l_{s_t}$  is the labor share.

Remarkably enough, as far as we know, there is no investigation of these issues in the literature. Such benign neglect contrasts with equivalent discussions in the

<sup>1</sup>In many economic literatures – e.g., in the real business cycle literature – it is often claimed that the presence of economic fluctuations (primarily at the business cycle frequency) is socially optimal: namely, efficient response to exogenous changes in the real economy, e.g., Cooley (1995).



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growth literature: since Ramsey (1928) the question of whether society saves or produces ‘too little’ (relative to a social planner) is fundamental (e.g., de La Grandville, 2012). Endogenous growth theory<sup>2</sup>, in addition, typically suggests that the socially optimal level of activity (in particular, R&D) exceeds the decentralized one, reflecting the absence of market distortions and externalities (e.g., Jones and Williams, 2000; Alvarez-Pelaez and Groth, 2005). But what of the labor share? Does ‘too little’ output in the decentralized economy imply too little labor share? *Ex ante*, it is by no means obvious. Given widespread interest in the labor share, this constitutes an important gap in our knowledge. Our paper addresses this gap.<sup>3</sup>

The structure of the paper is as follows. **Section 2** describes the model. We present a non-scale model of endogenous growth with two R&D sectors, giving rise to capital as well as labor augmenting innovations, drawing from the seminal contributions of Romer (1990) and Acemoglu (2003), and extending the model presented in Growiec et al. (2018). The model economy uses the Dixit-Stiglitz monopolistic competition setup and the increasing variety framework of the R&D sector. Both the social planner and decentralized allocations are solved for and compared. We see that the presence of markups arising from imperfect competition and externalities relating to R&D activity, are key points of comparison.

**Section 3** calibrates the model to US data. We assume that a range of long-run averages from US data correspond to the decentralized Balanced Growth Path (BGP) of the model. Thereafter, in **Section 4** we solve the BGP of each allocation and compare them. We list the channels and assumptions underlying the differences between both allocations. We also study the dependence of both long-run growth equilibria on model parameters and relate our results to Piketty’s “laws of capitalism”. Following Growiec et al. (2018) we also consider the dynamic properties of the model around the balanced growth path (both in the decentralized and optimal allocation) in terms of oscillatory dynamics.

We find that – assuming that factors are gross complements in production – the decentralized labor share is indeed socially suboptimal. The difference, moreover, is large: about 17% (11 pp). We describe the mechanisms which underly this wedge. For robustness, we also consider production characterized by Cobb Douglas as well as gross substitutes. In the latter case, and almost only in that case, the socially op-

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<sup>2</sup>For textbook overviews, see Barro and Sala-i-Martin (2003); Acemoglu (2009).

<sup>3</sup>Note, we do not only study the implications for the labor share, but also the growth rate, employment in the research sector, consumption and capital accumulation, etc. Furthermore, by providing a quantitative comparison between the decentralized and the socially optimal factor shares in an endogenous R&D-based growth setup, this paper contributes also to the literature on distributional effects of innovation (Aghion et al., 2015; Jones and Kim, 2018).

timal labor share falls below the decentralized one. However already for  $\sigma = 1.25$ , which constitutes a mild degree of gross substitutability, its value is counter-factually low (at around 0.5) and also associated with excessively high per capita growth rates.

**Section 5** concludes.

Note, however, that while making a first attempt at a new research question, we abstract from several issues. First, our concern is not about inequality per se. Although the labor share and inequality are clearly related, they are not interchangeable (Atkinson et al., 2011); an economy may well exhibit a socially-optimal factor income division yet still be characterized by considerable inequality. As, for example, if there are different skill characteristics in the labor force (and thus appreciable wage dispersion), or there is financial repression and rent seeking, and so on. Our concern therefore is somewhat more fundamental – namely, how would a social planner choose functional income shares. And, would that share be realistic<sup>4</sup> in terms of its mean value (and value relative to the decentralized optimum) and in terms of its volatility (again compared to the decentralized optimum and historical averages). Second, we do not discuss policy designs able to alleviate the discrepancy between the decentralized allocation and the social optimum.

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<sup>4</sup>By way of realism, consider another “optimal” rule in growth theory: namely the golden rule savings ratio which in standard form equates the optimal savings rate with the capital income share (which is usually around 30%), see de La Grandville (2012). With the exception of some Asian economies and for some particular periods, such values are highly counter factual.



## 2 Model

The framework is a generalization of Acemoglu (2003) with capital and labor augmenting R&D, building on the earlier induced innovation literature from Kennedy (1964) onwards as well as general innovation in monopolistic competition and growth literatures (Dixit and Stiglitz (1977), Romer (1990) and Jones (1999) and so on).

By ‘generalization’, we mean that we relax a number of features to make our conclusions more applicable to the studied question, as well as to correct for some counterfactual features (such as scale effects). Formally, (i) our model is non-scale: both R&D functions are specified in terms of percentages of population employed in either R&D sector; (ii) we also assume R&D workers are drawn from the same pool as production workers<sup>5</sup>; (iii) we assume more general R&D technologies which allow for mutual spillovers between both R&D sectors (cf. Li, 2000) and for concavity in capital augmenting technical change; (iv) in contrast to Acemoglu (2003), the BGP growth rate in our model depends on preferences via employment in aggregate production. The trade-off is due to drawing researchers from the same employment pool as production workers (a tradeoff not present in his model) and; (v) we use *normalized* CES production functions<sup>6</sup> which, importantly, ensures valid comparative static comparisons in the elasticity of factor substitution.

First, though, we start with the simpler benchmark of the social planner model.<sup>7</sup>

### 2.1 The Social Planner’s Problem

The social planner maximizes the representative household’s utility from discounted consumption, given standard CRRA preferences, (1), subject to the budget constraint (2) (i.e., the equation of motion of the aggregate per-capita capital stock), the ‘normalized’ production function, (3), the two R&D technologies (4)–(5) and the labor market

<sup>5</sup>Acemoglu (2003) assumes that labor supply in the production sector is inelastic and R&D is carried out by a separate group of “scientists” who cannot engage in production labor. Our assumption affects the tension between both R&D sectors by providing R&D workers with a third option, the production sector.

<sup>6</sup>Normalization essentially implies representing the production relations in consistent index number form. Its parameters then have a direct economic interpretation. Otherwise the parameters can be shown to be scale dependent (i.e., a circular function of  $\sigma$  itself, as well as a function of the implicit normalization points). Subscripts 0 denote the specific normalization points: geometric (arithmetic) averages for non-stationary (stationary) variables. See de La Grandville (1989); Klump and de La Grandville (2000); Klump and Preissler (2000) for the seminal theoretical contributions. In our case, normalization is essentially important since comparative statics on production function parameters are a key concern.

<sup>7</sup>It is simpler because solving under the social optimum we can impose symmetry directly and deal in terms of aggregates, see Bénassy (1998).

clearing condition, (6):<sup>8</sup>

$$\max \int_0^{\infty} \frac{c^{1-\gamma} - 1}{1-\gamma} e^{-(\rho-n)t} dt \quad s.t. \quad (1)$$

$$\dot{k} = y - c - (\delta + n)k - \zeta \dot{a}, \quad (2)$$

$$y = y_0 \left( \pi_0 \left( \lambda_b \frac{k}{k_0} \right)^{\xi} + (1 - \pi_0) \left( \frac{\lambda_a}{\lambda_{a0}} \frac{\ell_Y}{\ell_{Y0}} \right)^{\xi} \right)^{1/\xi} \quad (3)$$

$$\dot{\lambda}_a = A \left( \lambda_a \lambda_b^{\phi} x^{\eta_a} \ell_a^{\nu_a} \right), \quad (4)$$

$$\dot{\lambda}_b = B \left( \lambda_b^{1-\omega} x^{\eta_b} \ell_b^{\nu_b} \right) - d \lambda_b, \quad (5)$$

$$1 = \ell_a + \ell_b + \ell_Y, \quad (6)$$

where  $\gamma > 0$  is the inverse of the intertemporal elasticity of substitution,  $\rho > 0$  is the rate of time preference, and  $n > 0$  is the (exogenous) growth rate of the labor supply.

Parameter  $\xi$  is given by  $\xi = \frac{\sigma-1}{\sigma}$  where  $\sigma \in [0, \infty)$  is the elasticity of factor substitution. CES function (3) nests the linear, Cobb–Douglas, and Leontief forms respectively when  $\xi = 1, 0, -\infty$ . This parameter, important in many contexts<sup>9</sup>, turns out also to be critical in our analysis of the optimality and dynamics of the labor income share – relating, in particular, to whether factors are gross complements, i.e.,  $\sigma < 1$ , or gross substitutes.

In (2) and (3),  $y = Y/L$  and  $k = K/L$  (i.e., output and capital per capita), where  $L$  is total employment and  $\ell_a$  and  $\ell_b$  are the shares (or “research intensity”) employed in labor and capital augmenting R&D (i.e.,  $\lambda_a$  and  $\lambda_b$ ) respectively, with  $\ell_Y L = L_Y$ , etc. We assume that capital augmentation is subject to gradual decay at a rate  $d > 0$ , which mirrors susceptibility to obsolescence and embodied character of capital augmenting technologies, Solow (1960). This assumption is critical for the asymptotic constancy of unit capital productivity  $\lambda_b$  in the model, and thus for the existence of a BGP with purely labor augmenting technical change.

Factor augmenting innovations are created endogenously by the respective R&D sectors (Acemoglu, 2003), augmenting the technology menu by increasing the under-

<sup>8</sup>There are three control ( $c, \ell_a, \ell_b$ ) and three state variables ( $k, \lambda_a, \lambda_b$ ), in this optimization problem.

<sup>9</sup>The value of the elasticity of factor substitution has been shown to be a key parameter in many economic fields: e.g., the gains from trade (Saam, 2008); the strength of extensive growth (de La Grandville, 2016); multiple growth equilibria, development traps and indeterminacy (Azariadis, 1996; Klump, 2002; Kaas and von Thadden, 2003; Guo and Lansing, 2009), the responsive of investment and labor demand to various policy changes and shocks (Rowthorn, 1999; Chirinko, 2002) etc.

lying parameters  $\lambda_a, \lambda_b$ , as in (4) and (5). Parameters  $A$  and  $B$  capture the unit productivity of the labor and capital augmenting R&D process, respectively;  $\phi$  captures the spillover from capital to labor augmenting R&D;<sup>10</sup> and  $\omega$  measures the degree of decreasing returns to scale in capital augmenting R&D. By assuming  $\omega \in (0, 1)$  we allow for the “standing on shoulders” effect in capital augmenting R&D, albeit we limit its scope insofar as it is less than proportional to the existing technology stock (Jones, 1995).

Finally, R&D activity may be subject to duplication externalities; the greater the number of researchers searching for new ideas, the more likely is duplication. Thus research effort may be characterized by diminishing returns, Kortum (1993). This is captured by parameters  $\nu_a, \nu_b \in (0, 1]$ : the higher is  $\nu$  the lower the extent of duplication.<sup>11</sup>

Term  $x \equiv k(\lambda_b/\lambda_a)$  captures the technology-corrected degree of capital augmentation of the workplace. This “lab equipment” term will to be constant along the BGP. The long-term endogenous growth engine is located in the linear labor augmenting R&D equation. To fulfill the requirement of the existence of a BGP along which the growth rates of  $\lambda_a$  and  $\lambda_b$  are constant, we assume that  $\eta_b\phi + \eta_a\omega \neq 0$ .<sup>13</sup>

The last term in (2) captures a negative externality that arises from implementing new labor augmenting technologies, with  $\zeta \geq 0$ . Motivated by León-Ledesma and Satchi (2018), we allow for a non-negative cost of adopting new labor augmenting technologies: since workers (as opposed to machines) need to develop skills compatible with each new technology, it is assumed that there is an external capital cost of such technology shifts (training costs, learning-by-doing, etc.). We posit that new capital investments are diminished by  $\zeta\dot{a}$ , where  $\dot{a} = g\lambda_a \left(\frac{\pi}{\pi_0}\right)^{1/\alpha}$ ,  $g$  being the economic growth rate (Growiec et al., 2018).

## 2.2 Decentralized Allocation

The construction of the decentralized allocation draws from Romer (1990), Acemoglu (2003), Jones (2005), and Growiec et al. (2018). In particular, we use the Dixit and

<sup>10</sup>We assume  $\phi > 0$  indicating that more efficient use of physical capital also increases the productivity of labor augmenting R&D. Observe, there are *mutual* spillovers between both R&D sectors, with no prior restriction on their strength:  $\dot{\lambda}_a = A\lambda_a^{1-\eta_a}\lambda_b^{\phi+\eta_a}k^{\eta_a}\ell_a^{\nu_a}$  and  $\dot{\lambda}_b = B\lambda_a^{-\eta_b}\lambda_b^{1-\omega+\eta_b}k^{\eta_b}\ell_b^{\nu_b} - d\lambda_b$ .

<sup>11</sup>Observe that switching off all externalities and spillovers in (4)–(5) by setting  $d = \omega = \eta_a = \eta_b = 0$  and  $\nu_a = \nu_b = 1$  retrieves the original specification of R&D in Acemoglu (2003).<sup>12</sup> Moreover, compared with models which use Cobb Douglas production, equation (5) is akin to Jones’ (1995) formulation of the R&D sector, generalized by adding obsolescence and the lab equipment term. Thus, setting  $d = \eta_b = 0$  retrieves Jones’ original specification. And (4) is the same as in Romer (1990) but scale-free (it features a term in  $\ell_b$  instead of  $\ell_b \cdot L$ ) and with lab equipment and a direct spillover from  $\lambda_b$ ; setting  $\phi = \eta_a = 0$  retrieves the scale-free version of Romer (1990), cf. Jones (1999).

<sup>13</sup>All our qualitative results also go through for the special case  $\eta_a = \eta_b = 0$ , which fully excludes “lab equipment” terms in R&D. The current inequality condition is not required in such cases.

Stiglitz (1977) monopolistic competition setup and the increasing variety framework of the R&D sector. The general equilibrium is obtained as an outcome of the interplay between: households; final goods producers; aggregators of bundles of capital and labor intensive intermediate goods; monopolistically competitive producers of differentiated capital and labor intensive intermediate goods; and competitive capital and labor augmenting R&D firms.

### 2.2.1 Households

Analogous to the social planner's allocation (SP), we assume that the representative household maximizes discounted CRRA utility:

$$\max \int_0^{\infty} \frac{c^{1-\gamma} - 1}{1-\gamma} e^{-(\rho-n)t} dt \quad (7)$$

subject to the budget constraint:

$$\dot{v} = (r - \delta - n)v + w - c, \quad (8)$$

where  $v = V/L$  is the household's per-capita holding of assets,  $V = K + p_a \lambda_a + p_b \lambda_b$ . The representative household is the owner of all capital and also holds the shares of monopolistic producers of differentiated capital and labor intensive intermediate goods. Capital is rented at a net market rental rate equal to the gross rental rate less depreciation:  $r - \delta$ . Solving the household's optimization problem yields the familiar Euler equation:

$$\hat{c} = \frac{1}{\gamma}(r - \delta - \rho), \quad (9)$$

where  $\hat{c} = \dot{c}/c = g$  is the per-capita growth rate ("hats" denote growth rates.)

### 2.2.2 Final Goods Producers

The role of final goods producers is to generate the output of final goods (which are then either consumed by the representative household or saved and invested, leading to physical capital accumulation), taking bundles of capital and labor intensive intermediate goods as inputs. They operate in a perfectly competitive environment, where both bundles are remunerated at market rates  $p_K$  and  $p_L$ , respectively.

The final goods producers operate a normalized CES technology:

$$Y = Y_0 \left( \pi_0 \left( \frac{Y_K}{Y_{K0}} \right)^\xi + (1 - \pi_0) \left( \frac{Y_L}{Y_{L0}} \right)^\xi \right)^{\frac{1}{\xi}}. \quad (10)$$

The optimality condition implies that final goods producers' demand for capital and labor intensive intermediate goods bundles satisfies:

$$\frac{p_K}{p_L} = \frac{\pi}{1 - \pi} \frac{Y_L}{Y_K}, \quad (11)$$

where  $\pi = \pi_0 \left( \frac{Y_K}{Y_{K0}} \frac{Y_0}{Y} \right)^\xi$  is the elasticity of final output with respect to  $Y_K$ .

### 2.2.3 Aggregators of Capital and Labor Intensive Intermediate Goods

There are two symmetric sectors whose role is to aggregate the differentiated (capital or labor intensive) goods into the bundles  $Y_K$  and  $Y_L$  demanded by final goods producers. It is assumed that the differentiated goods are imperfectly substitutable (albeit gross substitutes). The degree of substitutability is captured by parameter  $\varepsilon \in (0, 1)$ :

$$Y_K = \left( \int_0^{N_K} X_{Ki}^\varepsilon di \right)^{\frac{1}{\varepsilon}}. \quad (12)$$

Aggregators operate in a perfectly competitive environment and decide upon their demand for intermediate goods, the price of which will be set by the respective monopolistic producers (discussed in the following subsection).

For capital intensive bundles, the aggregators maximize:

$$\max_{X_{Ki}} \left\{ p_K \left( \int_0^{N_K} X_{Ki}^\varepsilon di \right)^{\frac{1}{\varepsilon}} - \int_0^{N_K} p_{Ki} X_{Ki} di \right\}. \quad (13)$$

There is a continuum of measure  $N_K$  of capital intensive intermediate goods producers. Optimization implies the following demand curve:

$$X_{Ki} = x_K(p_{Ki}) = \left( \frac{p_{Ki}}{p_K} \right)^{\frac{1}{\varepsilon-1}} Y_K^{\frac{1}{\varepsilon}}. \quad (14)$$

Equivalent terms follow for labor intensive intermediate goods producers.

### 2.2.4 Producers of Differentiated Intermediate Goods

It is assumed that each of the differentiated capital or labor intensive intermediate goods producers, indexed by  $i \in [0, N_K]$  or  $i \in [0, N_L]$  respectively, has monopoly over its specific variety. It is therefore free to choose its preferred price  $p_{Ki}$  or  $p_{Li}$ . These firms operate a simple linear technology, employing either only capital or only labor.

For the case of capital intensive intermediate goods producers, the production function is  $X_{Ki} = K_i$ . Capital is rented at the gross rental rate  $r$ . The optimization problem is:

$$\max_{p_{Ki}} (p_{Ki} X_{Ki} - r K_i) = \max_{p_{Ki}} (p_{Ki} - r) x_K(p_{Ki}). \quad (15)$$

The optimal solution implies  $p_{Ki} = r/\varepsilon$  for all  $i \in [0, N_K]$ . This implies symmetry across all differentiated goods: they are sold at equal prices, thus their supply is also identical,  $X_{Ki} = \bar{X}_K$  for all  $i$ . Market clearing implies:

$$K = \int_0^{N_K} K_i di = \int_0^{N_K} X_{Ki} di = N_K \bar{X}_K \quad Y_K = N_K^{\frac{1-\varepsilon}{\varepsilon}} K. \quad (16)$$

The demand curve implies that the price of intermediate goods is linked to the price of the capital intensive bundle as in  $p_K = p_{Ki} N_K^{\frac{\varepsilon-1}{\varepsilon}} = \frac{r}{\varepsilon} N_K^{\frac{\varepsilon-1}{\varepsilon}}$ .

The *labor intensive* sector follows symmetrically:  $X_{Li} = L_{Yi}$ ,  $L_Y = \ell_Y L = \int_0^{N_L} L_{Yi} di$ , and  $p_{Li} = w/\varepsilon$ ,  $p_L = p_{Li} N_L^{\frac{\varepsilon-1}{\varepsilon}} = \frac{w}{\varepsilon} N_L^{\frac{\varepsilon-1}{\varepsilon}}$ , where  $w$  is the market wage rate.

Aggregating across all intermediate goods producers, we obtain that their total profits are equal to  $\Pi_K N_K = rK \left(\frac{1-\varepsilon}{\varepsilon}\right)$  and  $\Pi_L N_L = wL_Y \left(\frac{1-\varepsilon}{\varepsilon}\right)$  for capital and labor intensive goods respectively. Streams of profits per person in the representative household are thus  $\pi_K = \Pi_K/L$  and  $\pi_L = \Pi_L/L$ , respectively. Hence, the total remuneration channeled to the capital intensive sector equals  $p_K Y_K = \frac{r}{\varepsilon} K = rK + \Pi_K N_K$ , whereas the total remuneration channeled to the labor intensive sector equals  $p_L Y_L = \frac{w}{\varepsilon} L_Y = rL_Y + \Pi_L N_L$ . In equilibrium, factor shares then amount to,

$$\pi = \pi_0 \left(\frac{KY_0}{YK_0}\right)^\xi \left(\frac{N_K}{N_{K0}}\right)^{\xi \left(\frac{1-\varepsilon}{\varepsilon}\right)}, \quad (17)$$

$$1 - \pi = (1 - \pi_0) \left(\frac{Y_0 L_Y}{Y L_{Y0}}\right)^\xi \left(\frac{N_L}{N_{L0}}\right)^{\xi \left(\frac{1-\varepsilon}{\varepsilon}\right)}. \quad (18)$$

Incorporating all these choices into (10), and using the definitions  $\lambda_b = N_K^{\frac{1-\varepsilon}{\varepsilon}}$  and  $\lambda_a = N_L^{\frac{1-\varepsilon}{\varepsilon}}$  retrieves production function (3).

### 2.2.5 Capital and Labor Augmenting R&D Firms

The role of capital and labor augmenting R&D firms is to produce innovations which increase the variety of available differentiated intermediate goods, either  $N_K$  or  $N_L$ , and thus indirectly also  $\lambda_b$  and  $\lambda_a$ . Patents never expire, and patent protection is perfect. R&D firms sell these patents to the representative household which sets up a monopoly for each new variety. Patent price,  $p_b$  or  $p_a$ , which reflects the discounted stream of future monopoly profits, is set at the competitive market. There is free entry to R&D.

R&D firms employ labor only:  $L_a = \ell_a L$  and  $L_b = \ell_b L$  workers are employed in the labor and capital augmenting R&D sectors, respectively. There is also an externality from the total physical capital stock working through the “lab equipment” term in the R&D production function. Furthermore, the R&D firms perceive their production technology as linear in labor, while in fact it is concave due to duplication externalities.

Incorporating these assumptions and using the notion  $x \equiv \lambda_b k / \lambda_a$ , capital augmenting R&D firms maximize:

$$\max_{\ell_b} (p_b \dot{\lambda}_b - w \ell_b) = \max_{\ell_b} ((p_b Q_K - w) \ell_b), \quad (19)$$

where  $Q_K = B (\lambda_b^{1-\omega} x^{\eta_b} \ell_b^{\nu_b-1})$  is treated by firms as a constant in the steady state (Romer, 1990; Jones, 2005). Analogously, labor augmenting R&D firms maximize:

$$\max_{\ell_a} (p_a \dot{\lambda}_a - w \ell_a) = \max_{\ell_a} ((p_a Q_L - w) \ell_a), \quad (20)$$

where  $Q_L = A (\lambda_a \lambda_b^\phi x^{\eta_a} \ell_a^{\nu_a-1})$  is treated as exogenous.

Free entry into both R&D sectors implies  $w = p_b Q_K = p_a Q_L$ . Purchase of a patent entitles the holders to a per-capita stream of profits equal to  $\pi_K$  and  $\pi_L$ , respectively. While the production of any labor augmenting varieties lasts forever, there is a constant rate  $d$  at which production of capital intensive varieties becomes *obsolete*. This effect is external to patent holders and thus is not strategically taken into account when accumulating the patent stock.<sup>14</sup>

### 2.2.6 Equilibrium

We define the *decentralized equilibrium* as the collection of time paths of all the respective quantities:  $c, \ell_a, \ell_b, k, \lambda_b, \lambda_a, Y_K, Y_L, \{X_{Ki}\}, \{X_{Li}\}$  and prices  $r, w, p_K, p_L, \{p_{Ki}\}, \{p_{Li}\}, p_a, p_b$  such that: (1) households maximize discounted utility subject to their budget constraint; (2) profit maximization is followed by final-goods producers, aggregators and producers of capital and labor intensive intermediate goods, and capital and labor augmenting R&D firms; (3) the labor market clears:  $L_a + L_b + L_Y = (\ell_a + \ell_b + \ell_Y)L = L$ ; (4) the asset market clears:  $V = vL = K + p_a \lambda_a + p_b \lambda_b$ , where assets have equal returns:  $r - \delta = \frac{\pi_L}{p_a} + \frac{\dot{p}_a}{p_a} = \frac{\pi_K}{p_b} + \frac{\dot{p}_b}{p_b} - d$ ; and, finally (5), such that the aggregate capital stock satisfies  $\dot{K} = Y - C - \delta K - \zeta \dot{a}L$ , where the last term is an externality term (as previously discussed).

<sup>14</sup>In other words, by solving a static optimization problem, capital augmenting R&D firms do not take the dynamic (external) obsolescence effect into account.



## 2.3 Solving for the Social Planner Allocation

In this section, we first solve analytically for the BGP of the social planner allocation of our endogenous growth model and then linearize the implied dynamical system around the BGP.

### 2.3.1 Balanced Growth Path

Any neoclassical growth model can exhibit balanced growth only if technical change is purely labor augmenting or if production is Cobb Douglas, Uzawa (1961). This conclusion holds here too. Hence, once we presume a CES production function, the analysis of dynamic consequences of technical change, which is not purely labor augmenting, must be done outside the BGP.

Along the BGP, we obtain the following growth rate of key model variables:

$$g = \hat{\lambda}_a = \hat{k} = \hat{c} = \hat{y} = A(\lambda_b^*)^\phi (x^*)^{\eta_a} (\ell_a^*)^{\nu_a}, \quad (21)$$

where stars denote steady-state values. Hence, ultimately long-run growth is driven by labor augmenting R&D. This can be explained by the fact that labor is the only non-accumulable factor in the model, it is complementary to capital along the aggregate production function, and the labor augmenting R&D equation is linear with respect to  $\lambda_a$ . The following variables are constant along the BGP:  $y/k$ ,  $c/k$ ,  $\ell_a$ ,  $\ell_b$  and  $\lambda_b$  (i.e., asymptotically there is no capital augmenting technical change).

### 2.3.2 Euler Equations

Having set up the Hamiltonian and computed its derivatives, the following Euler equations are obtained for the SP:<sup>15</sup>

$$\hat{c} = \frac{1}{\gamma} \left( \frac{y}{k} \left( \pi + \frac{1-\pi}{\ell_Y} \left( \frac{\eta_a \ell_a}{\nu_a} + \frac{\eta_b \ell_b}{\nu_b} \right) \right) - \delta - \rho \right), \quad (22)$$

$$\varphi_1 \hat{\ell}_a + \varphi_2 \hat{\ell}_b = Q_1, \quad (23)$$

$$\varphi_3 \hat{\ell}_a + \varphi_4 \hat{\ell}_b = Q_2, \quad (24)$$

<sup>15</sup>A sufficient condition for all transversality conditions to be satisfied in the social optimum (as well as in the decentralized equilibrium) is that  $(1-\gamma)g + n < \rho$ .

where

$$\varphi_1 = \nu_a - 1 - (1 - \xi)\pi \frac{\ell_a}{\ell_Y}, \quad (25)$$

$$\varphi_2 = -(1 - \xi)\pi \frac{\ell_b}{\ell_Y} \quad (26)$$

$$\varphi_3 = -(1 - \xi)\pi \frac{\ell_a}{\ell_Y} \quad (27)$$

$$\varphi_4 = \nu_b - 1 - (1 - \xi)\pi \frac{\ell_b}{\ell_Y}, \quad (28)$$

$$Q_1 = -\gamma\hat{c} - \rho + n + \hat{\lambda}_a \left( \frac{\ell_Y \nu_a}{\ell_a} + 1 - \eta_a - \eta_b \frac{\ell_b \nu_a}{\ell_a \nu_b} \right) - \phi \hat{\lambda}_b + ((1 - \xi)\pi - \eta_a)\hat{x} \quad (29)$$

$$Q_2 = -\gamma\hat{c} - \rho + n + \hat{\lambda}_a + \hat{\lambda}_b \left( \frac{\pi}{1 - \pi} \frac{\ell_Y \nu_b}{\ell_b} + (\phi + \eta_a) \frac{\nu_b \ell_a}{\nu_a \ell_b} + \eta_b \right) + ((1 - \xi)\pi - \eta_b)\hat{x} + d \left( \frac{\pi}{1 - \pi} \frac{\ell_Y \nu_b}{\ell_b} + (\phi + \eta_a) \frac{\nu_b \ell_a}{\nu_a \ell_b} - \omega + \eta_b \right). \quad (30)$$

### 2.3.3 Steady State and Linearization of the Transformed System

The above Euler equations and dynamics of state variables are then rewritten in terms of *stationary* variables which are constant along the BGP, i.e., in coordinates:  $u = (c/k), \ell_a, \ell_b, x, \lambda_b$ , and with auxiliary variables  $z = (y/k), \pi, g$ . The full steady state of the transformed system is listed in Appendix A.1. This non-linear system of equations is solved numerically, yielding a unique steady state of the de-trended system, and thus a unique BGP of the model in original variables. All further analysis of the social planner allocation is based on the (numerical) linearization of the 5-dimensional dynamical system of equations (22)–(24), (2) and (5), taking the BGP equality (21) as given.

## 2.4 Solving for the Decentralized Allocation

When solving for the decentralized allocation, we broadly follow the steps carried out in the case of the social planner allocation. We first solve analytically for the BGP of our endogenous growth model and then linearize the implied dynamical system around the BGP.

### 2.4.1 Balanced Growth Path

Along the BGP, we obtain the following growth rate of the key model variables:

$$g = \hat{k} = \hat{c} = \hat{y} = \hat{w} = \hat{p}_b = \hat{p}_{Li} = \hat{\lambda}_a = A(\lambda_b^*)^\phi (x^*)^{\eta_a} (\ell_a^*)^{\nu_a}. \quad (31)$$

The following quantities are constant along the BGP:  $y/k, c/k, \ell_a, \ell_b, Y_K/Y, Y_L/Y$  and  $\lambda_b$  (again, note, asymptotically, the absence of capital augmenting technical change).

The following prices are also constant along the BGP:  $r, p_a, p_K, p_L, \{p_{K_i}\}$ .

### 2.4.2 Euler Equations

The decentralized equilibrium is associated with the following Euler equations describing the first-order conditions:

$$\hat{c} = \frac{1}{\gamma} \left( \varepsilon \pi \frac{y}{k} - \delta - \rho \right), \quad (22')$$

$$\varphi_1 \hat{\ell}_a + \varphi_2 \hat{\ell}_b = \tilde{Q}_1, \quad (23')$$

$$\varphi_3 \hat{\ell}_a + \varphi_4 \hat{\ell}_b = \tilde{Q}_2, \quad (24')$$

where

$$\tilde{Q}_1 = -\varepsilon \pi \frac{y}{k} + \delta + \hat{\lambda}_a \frac{\ell_Y}{\ell_a} - \phi \hat{\lambda}_b + ((1 - \xi)\pi - \eta_a) \hat{x} \quad (29')$$

$$\tilde{Q}_2 = -\varepsilon \pi \frac{y}{k} + \delta + \hat{\lambda}_a + (\hat{\lambda}_b + d) \left( \frac{\pi}{1 - \pi} \frac{\ell_Y}{\ell_b} \right) - \hat{\lambda}_b (1 - \omega) - d + ((1 - \xi)\pi - \eta_b) \hat{x} \quad (30')$$

and  $\varphi_1$  through  $\varphi_4$  are defined as in (25)–(28). The full steady state of the transformed system is listed in Appendix A.2. All further analysis of the decentralized allocation is based on the (numerical) linearization of the 5-dimensional dynamical system of equations (22')–(24'), (2) and (5), taking the BGP equality (31) as given.

### 2.4.3 Departures from the Social Optimum

Departures of the decentralized allocation from the optimal one can be tracked back to specific assumptions regarding the information structure of the decentralized allocation. Those differences are the following:

1. In the consumption Euler equation, comparing equations (22) with (22'), the term  $\frac{y}{k} \left( \pi + \frac{1 - \pi}{\ell_Y} \left( \frac{\eta_a \ell_a}{\nu_a} + \frac{\eta_b \ell_b}{\nu_b} \right) \right)$  is replaced by  $\varepsilon \pi \frac{y}{k}$ . This is due to two effects:
  - (a) in contrast to the social planner, markets fail to account for the external effects of physical capital on R&D activity via the lab equipment terms (with respective elasticities  $\eta_b$  and  $\eta_a$ );
  - (b)  $\varepsilon$  appears in the decentralized allocation due to imperfect competition in the labor and capital augmenting intermediate goods sectors.
2. In the Euler equations for  $\ell_a$  and  $\ell_b$  (equations (23), (24), (23'), (24')) the shadow price of physical capital  $\hat{c} - \rho + n$  is replaced by its market price  $r - \delta = \varepsilon \pi \frac{y}{k} - \delta$  which accounts for markups arising from imperfect competition.

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3. In the Euler equation for  $\ell_a$ , the term  $\left(\frac{\ell_Y \nu_a}{\ell_a} + 1 - \eta_a - \eta_b \frac{\ell_b \nu_a}{\ell_a \nu_b}\right)$  is replaced by  $\frac{\ell_Y}{\ell_a}$ .

This is due to two effects:

- (a)  $\nu_a$  is missing because markets fail to internalize the labor augmenting R&D duplication effects when  $\nu_a < 1$ ;
- b) the latter two components are missing because markets fail to account for the external effects of accumulating knowledge on future R&D productivity. These effects are included in the shadow prices of  $\lambda_a$  and  $\lambda_b$  in the social planner allocation but not in their respective market prices.

4. Analogously, in the Euler equation for  $\ell_b$ , the term given by,

$\frac{\ell_Y \nu_b}{\ell_b} \frac{\pi}{1-\pi} + (1 - \omega) + \eta_b + (\phi + \eta_a) \frac{\ell_a \nu_b}{\ell_b \nu_a}$  is replaced by  $\frac{\ell_Y}{\ell_b} \frac{\pi}{1-\pi}$ . The same reasoning follows as per point 3.

### 3 Calibration of the Model

The calibration for the decentralized model is listed in **Table 1**. We assume that a range of long-run averages from US data correspond to the decentralized BGP of the model (1929-2015). Doing so allows us to calibrate the rates of economic and population growth, capital productivity and income share, and the consumption-to-capital ratio. Likewise, we assign CES normalization parameters to match US long-run averages for factor income shares (we adjust the payroll share by proprietors' income, Mućk et al. (2018)). This implies an average labor share of 0.67.

Next, we turn to the elasticity of substitution between labor and capital ( $\sigma$ ) which is the fundamental economic parameter in our analysis. We calibrate factors to be gross complements, i.e.,  $\sigma < 1$ . This choice stems out from a fact that most of the empirical studies for the US aggregated production function document that the  $\sigma$  is systematically below unity (Chirinko, 2008; Klump et al., 2012).<sup>16</sup> Most of the empirical evidence exploiting time series variation for other countries also implies  $\sigma < 1$  (McAdam and Willman, 2013; Mućk, 2017).

However, the literature based predominantly on cross-country variation is rather inconclusive about the magnitude of  $\sigma$ . On one hand, several papers (Piketty, 2014; Piketty and Zucman, 2014; Karabarbounis and Neiman, 2014) employ gross substitutes. On the other, recent studies exploiting macro panels and allowing for factor augmentation in the supply-side system approach tend to return to the conclusion of gross complementarity (Villacorta, 2018; Mućk, 2017). Given this, we also consider  $\sigma < 1$  as a benchmark, but  $\sigma > 1$  is used in our robustness exercises.

Next, given this, four identities included in the system (see appendix equations A.42–A.50) drive the calibration of other parameters in a model-consistent manner:  $\ell_Y^*$ ,  $r^*$ ,  $\lambda_b^*$ ,  $x^*$  and  $\varepsilon$ . Employment in final production is also set in a model-consistent manner. In the absence of any other information, we agnostically assume that the share  $1 - \ell_Y^*$  is split equally between employment in both R&D sectors. For the model-consistent value of  $\ell_Y^*$ , this formula leads to values close to those typically considered for the non-routine cognitive occupational group (e.g., Jaimovich and Siu (2012), using BLS data, show this ratio to be between 29% and 38% (over 1982-2012)).

Regarding the preference parameters, for the (inverse) intertemporal elasticity of substitution and the rate of time preference, we use values typically found in the liter-

<sup>16</sup>For instance, the seminal Arrow et al. (1961) paper found an aggregate elasticity over 1909-1949 of 0.57 which, at the aggregate level, is confirmed by Antràs (2004). More recently, Klump et al. (2007) reported  $\hat{\sigma} \approx 0.7$ . The gross complementarity between factors is also confirmed at the industry (Herrendorf et al., 2015; Chirinko and Mallick, 2017) and firm level (Oberfield and Raval, 2018). Importantly, the elasticity uncovered is found systematically below unity even if more flexible functional forms of aggregate production function are considered (Growiec and Mućk, 2016).

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ature. For the duplication externalities, we agnostically assume  $\nu_a = \nu_b = 0.75$  following the (albeit single R&D sector) value in Jones and Williams (2000). The technology augmenting term  $\lambda_b^*$  is set in a model-consistent manner, normalized to unity.

The final step is to assign values to the remaining parameters, in particular the technological parameters of the R&D equations. We do this by solving the four remaining equations in system (A.42)–(A.50) with respect to the remaining parameters, see **Table B.1**. Given this benchmark calibration, the steady state is a saddle point.

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TABLE 1: BASELINE CALIBRATION: PRE-DETERMINED PARAMETERS (DECENTRALIZED ALLOCATION)

Parameter	Symbol	Value	Source/Target
<i>Income and Production</i>			
GDP Per-Capita Growth	$g$	0.0171	geometric average of gross growth rates
Population Growth rate	$n$	0.0153	geometric average of gross growth rates
Labor in Aggregate Production	$\ell_{Y0}, \ell_Y^*$	0.5394	$\frac{0.5(\gamma+\frac{\rho}{g})}{1+0.5(\gamma+\frac{\rho}{g})}$
Capital Productivity	$z_0, z^*$	0.3442	geometric average
Consumption-to-Capital	$u^*$	0.2199	geometric average
Capital Income Share	$\pi_0, \pi^*$	0.3260	arithmetic average
Depreciation	$\delta$	0.0600	Caselli (2005)
Factor Substitution Parameter	$\xi$	-0.4286	$\Rightarrow \sigma = 0.7$ , Klump et al. (2007)
Net Real Rate of Return	$r^* - \delta$	0.0499	$r^* - \delta = \gamma g + \rho$
Substitutability Between Intermediate Goods	$\varepsilon$	0.9793	$\varepsilon = \frac{r^*}{\pi^* z^*}$
<i>R&amp;D Sectors</i>			
R&D Duplication Parameters	$\nu_a = \nu_b$	0.7500	Jones and Williams (2000)
Technology Augmenting Terms	$\lambda_{a0}, \lambda_{b0}$	1.0000	see text
Technology Augmenting Terms	$\lambda_b^*$	1.0000	$\lambda_b^* = \lambda_{b0} \frac{z^*}{z_0} \left(\frac{\pi^*}{\pi_0}\right)^{\frac{1}{\xi}}$
Employment Share in R&D sectors	$\ell_Y^a, \ell_Y^b$	0.2033	$\ell_a^* = \ell_b^*$ for $\ell_a^* + \ell_b^* = 1 - \ell_Y^*$
Lab-Equipment Term <sup>†</sup>	$x_0, x^*$	61.7900	$x^* = x_0 \frac{\ell_Y^*}{\ell_{Y0}} \left(\frac{1}{1-\pi_0} \left(\frac{z^* \lambda_{b0}}{z_0 \lambda_b^*}\right)^\xi - \frac{\pi_0}{1-\pi_0}\right)^{-1/\xi}$
<i>Preferences</i>			
Inverse Intertemporal Elasticity of Substitution	$\gamma$	1.7500	Barro and Sala-i-Martin (2003)
Time Preference	$\rho$	0.0200	Barro and Sala-i-Martin (2003)

Notes: <sup>†</sup>  $x_0 = \frac{\lambda_{a0} \lambda_{b0}}{\lambda_{a0}} = 61.79$ .



## 4 Is the Decentralized Labor Share Socially Optimal?

Given the model set up and its benchmark calibration, we can now come to our central question: is the decentralized labor share socially optimal? In **Table 2**, columns (1) and (2) show the decentralized allocation (DA) and social planner (SP) outcomes for our benchmark calibration; (3) and (4), considered later, alternatively impose Cobb–Douglas and gross substitutes.

**TABLE 2: BGP COMPARISON UNDER THE BASELINE CALIBRATION**

Variable	Symbol	(1)	(2)	(3)	(4)
		DA	SP		
			Baseline $\sigma = 0.7$ $\xi = -0.43$	C–D $\sigma = 1$ $\xi = 0$	Piketty $\sigma = 1.25$ $\xi = 0.20$
Output growth rate	$g$	0.0171	0.0339	0.0425	0.0581
Consumption-to-capital ratio	$u^*$	0.2199	0.1628	0.1180	0.0856
Capital productivity	$z^*$	0.3442	0.3071	0.2832	0.2743
Employment in production	$\ell_Y^*$	0.5934	0.4385	0.4160	0.3854
Employment in labor augmenting R&D	$\ell_a^*$	0.2033	0.2575	0.2447	0.2240
Employment in capital augmenting R&D	$\ell_b^*$	0.2033	0.3040	0.3393	0.3906
Relative Share	$\ell_a^*/\ell_b^*$	1	0.8470	0.7212	0.5735
Labor income share	$1 - \pi^*$	0.6739	0.7854	0.6739	0.5243
Relative to DA (%)	$\frac{1 - \pi^{*DA}}{1 - \pi^*} - 1$	0	0.1655	0	-0.2220
Capital income share	$\pi^*$	0.3261	0.2146	0.3261	0.4757
Net real rate of return	$r^* - \delta$	0.0499	0.0059	0.0323	0.0704
Capital augmenting technology	$\lambda_b^*$	1.0000	2.3696	3.3162	5.2600
Lab equipment	$x^*$	61.7900	173.3363	342.7082	928.9625

The BGP of the decentralized solution features lower growth, lower R&D activity, but higher consumption ( $u$  is higher). Moreover, with lower growth, capital costs are cheaper and the net real rate of return of capital is higher, and capital productivity is accordingly higher. That there should be a growth differential in favor of the social planner is straightforward; it follows from our discussion in **Section 2.4.3**: due to externalities, there is insufficient R&D activity and thus lower growth. But the comparison in terms of the labor share is perhaps less obvious. In fact we see the striking result that the labor share in the social optimum is around 17% (11 pp) *above* the decentralized allocation.

To understand why, let us decompose the capital income share,  $\pi$ , in the following two ways (recalling that  $1 - \pi$  is the labor income share):

$$\frac{\pi}{\pi_0} = \left( \frac{\lambda_b k}{k_0} \right)^\xi \left( \frac{y}{y_0} \right)^{-\xi} \Rightarrow \hat{\pi} = \xi(\hat{\lambda}_b + \hat{k} - \hat{y}), \quad (32)$$

$$\frac{\pi}{1 - \pi} = \frac{\pi_0}{1 - \pi_0} \left( \frac{x \ell_{Y0}}{x_0 \ell_Y} \right)^\xi \Rightarrow \hat{\pi} = \xi(1 - \pi)(\hat{x} - \hat{\ell}_Y). \quad (33)$$

Equation (32) shows that under gross complementarity ( $\sigma < 1$ , or equivalently  $\xi < 0$ ), the capital share decreases with inverse capital productivity and with capital augmentation (i.e, the capital augmenting technology improvements are “labor biased”).

Equation (33), in turn, follows from the definition of the aggregate production function and the “lab equipment” term  $x$ . Given  $\hat{\ell}_Y \equiv -\left(\frac{\ell_a}{\ell_Y} \hat{\ell}_a + \frac{\ell_b}{\ell_Y} \hat{\ell}_b\right)$ , the dynamics of employment in the goods sector are equal to the inverse of the dynamics of total R&D employment. It then follows that dynamics of the labor share are *uniquely determined* by the sum of the dynamics of the lab equipment component and R&D employment. As before, the sign of this relationship depends upon the substitution elasticity: if  $\xi < 0$  then increases in R&D intensity reduce  $\pi$ , and thus increase the labor share, and vice versa.

Comparing the decentralized and the social planner’s allocation through the lens of (32), we observe that the large difference in factor shares at the BGP is driven almost exclusively by the difference in the level of capital augmentation  $\lambda_b^*$ . This result suggests that technical change is quantitatively more important for explaining labor share developments than the share of the capital stock in output.

Equivalently, by (33), this large difference in the degree of capital augmentation shows up in the lab equipment term  $x^*$ . It is also strengthened by the discrepancy in employment in final production  $\ell_Y^*$ , which is higher in the decentralized allocation because the planner devotes more resources to (both types of) R&D. Thanks to this, coupled with relatively more saving, it achieves faster growth at the BGP but with a lower consumption-to-capital ratio and a lower rate of return to capital. All of these make for a higher labor share in the optimal allocation.

#### 4.1 Impact of Parameter Variation on the Equilibrium Labor Share

The results just discussed hold for the benchmark calibration. Accordingly, we now consider sensitivity to deviations from that calibration. **Figure 2** presents the impact of varying selected model parameters, holding others constant, on the BGP level of the labor share.

Essentially all panels can be interpreted through the lens of equations (32) and (33). As agents become less patient (higher  $\rho$ ), R&D intensity falls, as does the la-

bor share. Similar reasoning pertains to the inverse intertemporal elasticity of substitution  $\gamma$ . That  $\frac{\partial(1-\pi)}{\partial\eta_b} > 0$  arises from the usual property that, under our gross-complements benchmark, improvements in capital augmenting technical change are labor biased; analogously  $\frac{\partial(1-\pi)}{\partial\eta_a} < 0$ . Likewise, we have under gross complements:  $\frac{\partial(1-\pi)}{\partial\nu_a} > 0$ ,  $\frac{\partial(1-\pi)}{\partial\nu_b} < 0$ . If capital depreciates faster, the capital (labor) share rises (falls).

Finally, we see that under gross substitutes,  $\sigma > 1$  ( $\xi > 0$ ), the DA labor share *exceeds* that of the SP.<sup>17</sup> We discuss this case further below, but it is straightforward to motivate, since the previously-discussed mechanisms go into reverse; technical improvements tend to be capital biased.<sup>18</sup> A more extensive study of the dependence of both BGPs on key model parameters ( $\rho, \gamma, \nu_b, \eta_b$ ) is included in **Figure C.1–Figure C.3**. And the equivalent figure for the gross substitutes case is given in **Figure C.4**. They are essentially a mirror image of our benchmark gross complements case.

## 4.2 Impact of Parameter Variation on the Equilibrium Growth Rate

So far we have confirmed the received wisdom that  $g^{DA}$  is socially suboptimal. This appears to be generally true in our model, regardless of its parametrization. Focusing on a reasonable parameter support, and assuming gross complements (see **Figure 3**), we can however identify a few credible cases where the difference between the two growth rates becomes small:

1. A higher  $\rho$  (i.e., more impatience for current consumption), implies less capital and R&D accumulation and lower equilibrium growth than otherwise. If  $\rho^{SP} \gg \rho^{DA}$  then  $g^{SP} \rightarrow_+ g^{DA}$ .<sup>19</sup>
2. If the consumption smoothing motive is sufficiently weak ( $\gamma$  high) then  $g^{SP} \rightarrow_+ g^{DA}$ .
3. If the lab equipment exponents are weak,  $\nu_a \rightarrow 0$  or  $\nu_b \rightarrow 0$ , then they attenuate the engine of long run growth in the R&D equations and thus pull both  $g^{SP}$  and  $g^{DA}$  down.

Finally, note, departing from gross substitutes, we see the dramatic result that as  $\sigma$  (or equivalently  $\xi$ ) increases, the gap  $g^{SP} - g^{DA}$  exponentially widens; conversely it

<sup>17</sup>The second, albeit weaker case is when  $\eta_b$ , the exponent on the lab equipment term in capital augmenting technical progress, becomes very small.

<sup>18</sup>Note that the lack of dependence of the BGP on  $\xi$  in the decentralized allocation follows from CES normalization (Klump and de La Grandville, 2000), coupled with the fact that we have calibrated the normalization constants to the BGP of the decentralized allocation.

<sup>19</sup>See also **Figure C.1**.

narrows as substitution possibilities tend to zero (the Leontief case). We explore this in the next section.

### 4.3 Impact of Elasticity of Substitution Variation on the BGP

Although we regard the gross complements case to be the more empirically relevant (at least for the aggregate economy), we also investigate the Cobb Douglas and gross substitutes case. Accordingly, the SP is solved anew and presented under columns (3) and (4) of **Table 2**, respectively. The effect of a continuous variation in the substitution elasticity is graphed in **Figure 4**.

Both alternative parametrizations are markedly more growth friendly. Per-capita output grows at the counterfactual rate of around 4 – 6%, exceeding both the previous SP and DA by a large margin, with an inflection point at  $\xi \approx 0.25$ , after which it shoots through the roof. That steady state per-capita growth is an increasing function of the substitution elasticity, though, is to be expected. Intuitively, easier factor substitution – by staving off diminishing returns – can prolong extensive growth (i.e., scarce factors can be substituted by abundant ones). The formal proof of this<sup>20</sup> can be though related through the properties of the normalized CES function as a General Mean function.<sup>21</sup>

The consequences for labor's share of income, though, are dire. With gross factor substitutability,<sup>22</sup> the arguments of the previous section shift into reverse. Capital improvements are capital biased, and the incentives for capital accumulation are accordingly far higher in this regime. Hence the labor share declines with  $\sigma$  (or equivalently  $\xi$ ).

It should also be emphasized that a balanced growth path does not exist in our model under sufficiently strong factor substitutability. Gross substitutability, as such, implies that Inada conditions at infinity are violated: the marginal product of per-capita capital (MPK) remains bounded *above* zero as the capital stock goes to in-

<sup>20</sup>See the discussion in Pitchford (1960) and the subsequent discussions in de La Grandville (1989); Klump and de La Grandville (2000); Klump and Preissler (2000); Palivos and Karagiannis (2010).

<sup>21</sup>A General Mean of order  $\xi$  is defined as  $M(\xi) = \left[ \sum_{i=1}^n f_i x_i^\xi \right]^{\frac{1}{\xi}}$  where  $x_1, \dots, x_n$  are positive numbers (of the same dimension) and where the weights  $f_1, \dots, f_n$  sum to unity. A General Mean is an increasing function of its order: the mean of order  $\xi$  of the positive values  $x_i$  with weights  $f_i$  is a strictly increasing function in  $\xi$  unless all the  $x_i$  are equal. This is often known as the de La Grandville hypothesis after de La Grandville (1989). He attributed that the rapid growth in East Asian countries to a high substitution factor elasticity value in their industrial sectors, and their high savings rate. This echoes the argument of Hicks (1932) that larger substitution value entails high transformation rates between sectors of different factor intensity. When one activity contracts to the benefit of another, the production increase in the second sector can be made larger if the substitution elasticity is high.

<sup>22</sup>In the Cobb–Douglas case of  $\xi = 0$ , factor shares are constant and at their predetermined sample average. Thus  $\pi|_{\xi=0} = \pi_0$ .

finity. But then there is still the question whether the lower bound of MPK, multiplied by the savings rate, is high enough to exceed the capital depreciation rate. If so, and this happens only when  $\sigma$  exceeds a certain threshold  $\bar{\sigma} > 1$ , endogenous growth driven by capital accumulation appears (Jones and Manuelli, 1990; Palivos and Karagiannis, 2010). Combined with the existing growth engine of our model – labor augmenting R&D – both sources of growth then lead to hyper-exponential, explosive growth. Then, even with diminishing returns to factors, capital intensity grows without bounds, labor becomes inessential in production, and hence the capital income share tends to unity. We rule such cases out of our analysis (see **Figure 2–Figure 4**).

#### 4.4 Relation to Piketty’s Laws

As our model endogenizes both economic growth and factor shares, it constitutes an appropriate framework for studying the “Fundamental Laws of Capitalism” formulated by Piketty (2014), i.e. (i) that the degree of capital deepening  $K/Y$  goes up whenever the economic growth rate  $g$  goes down, and (ii) that the capital share  $\pi$  goes up whenever the growth rate  $g$  goes down. Our setup has the advantage over Piketty’s in that all three variables are endogenous, and hence one can legitimately observe whether changing some parameters implies relations that are in line with Piketty’s claims (i)–(ii), or go in the opposite direction.

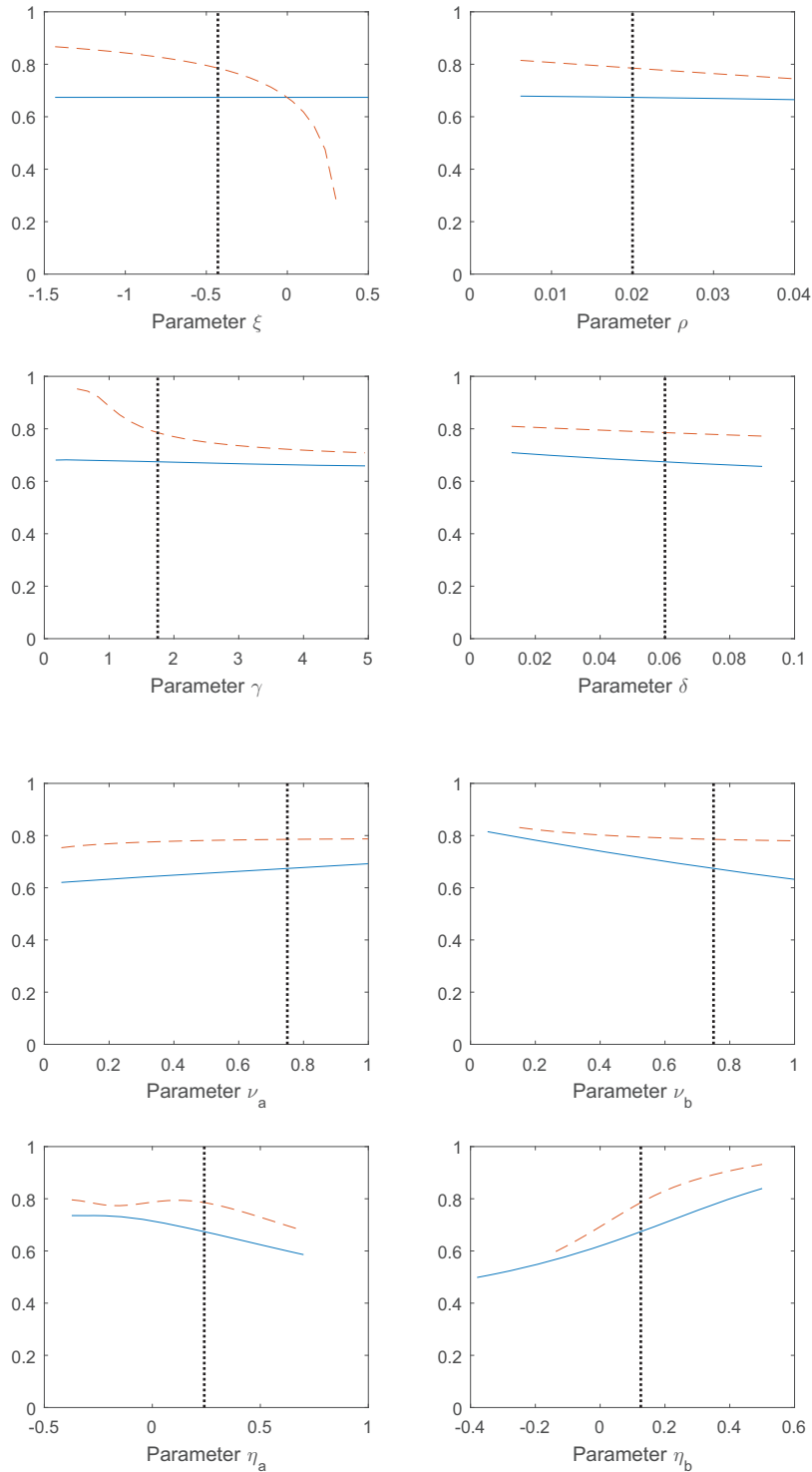
First, taking Piketty’s claims (i)–(ii) together logically implies that  $K/Y$  and the capital share  $\pi$  are positively correlated, suggesting that capital and labor should be gross substitutes ( $\sigma > 1$ ), see equation (32). This is a widely recognized issue with Piketty’s claims (see e.g. Rognlie (2015); Oberfield and Raval (2018)). In our baseline parametrization, we assume gross complements instead.

Second, inspection of **Figure 5** reveals that under the baseline calibration, both in the decentralized equilibrium and the social planner allocation:

- when households become more patient ( $\rho$  goes down) or more willing to substitute consumption intertemporally ( $\gamma$  goes down), only law (ii) holds: the growth rate  $g$  goes up, the  $K/Y$  ratio goes up, and the capital share  $\pi$  goes down;
- when the lab equipment exponent  $\nu_b$  in capital augmenting R&D goes up, both laws are verified: the growth rate  $g$  goes down, the  $K/Y$  ratio goes up, and the capital share  $\pi$  goes up.

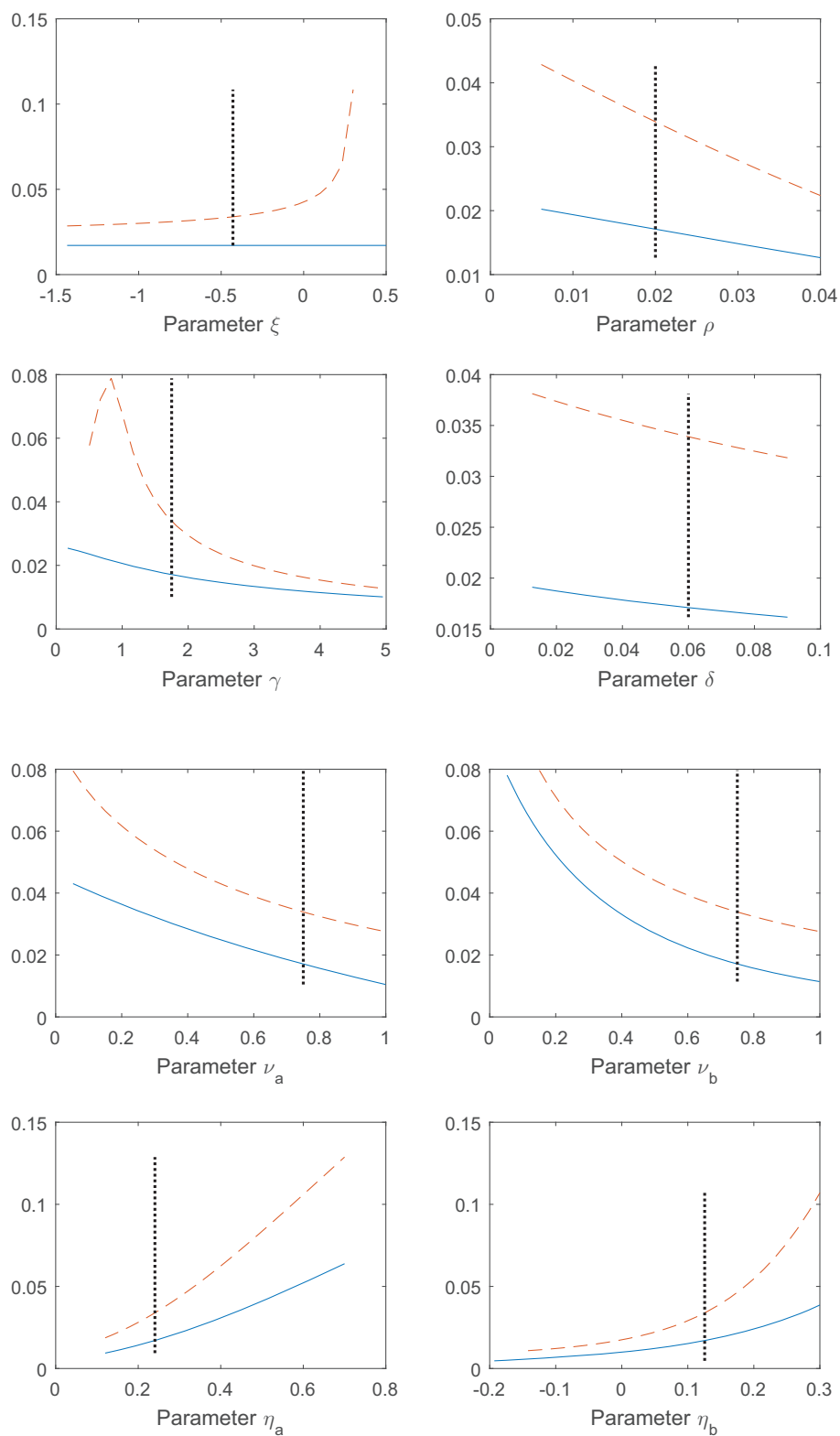
Third, we find (**Figure 4**) that as the elasticity of substitution goes up, the optimal growth rate  $g$  goes up hand in hand with the capital share  $\pi$  and the  $K/Y$  ratio. In such case, *both* of Piketty’s laws are violated.

**FIGURE 2: DEPENDENCE OF EQUILIBRIUM LABOR SHARE ON MODEL PARAMETERS.**



**Notes:**  $1 - \pi$  on vertical axis; corresponding parameter support on the horizontal axis. Social planner allocation (dashed lines), decentralized equilibrium (solid lines). The vertical dotted line in each graph represents the baseline calibrated parameter value.

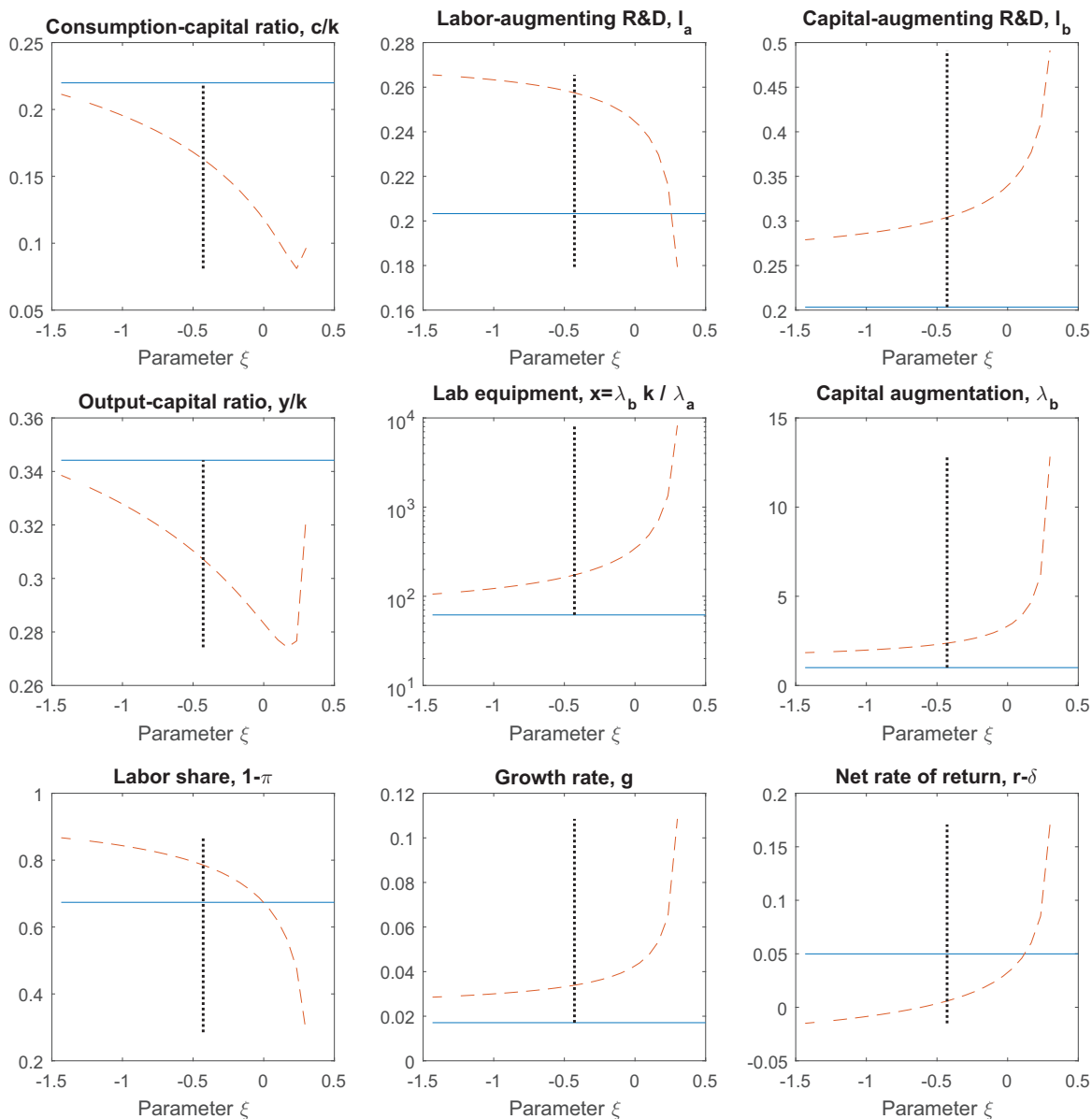
**FIGURE 3: DEPENDENCE OF EQUILIBRIUM GROWTH ON MODEL PARAMETERS.**



**Notes:** The real economic growth rate  $g$  on vertical axis; corresponding parameter support on the horizontal axis. Social planner allocation (dashed lines), decentralized equilibrium (solid lines). The vertical dotted line in each graph represents the baseline calibrated parameter value.



FIGURE 4: DEPENDENCE OF BGP ON ELASTICITY OF SUBSTITUTION, DA vs. SP.



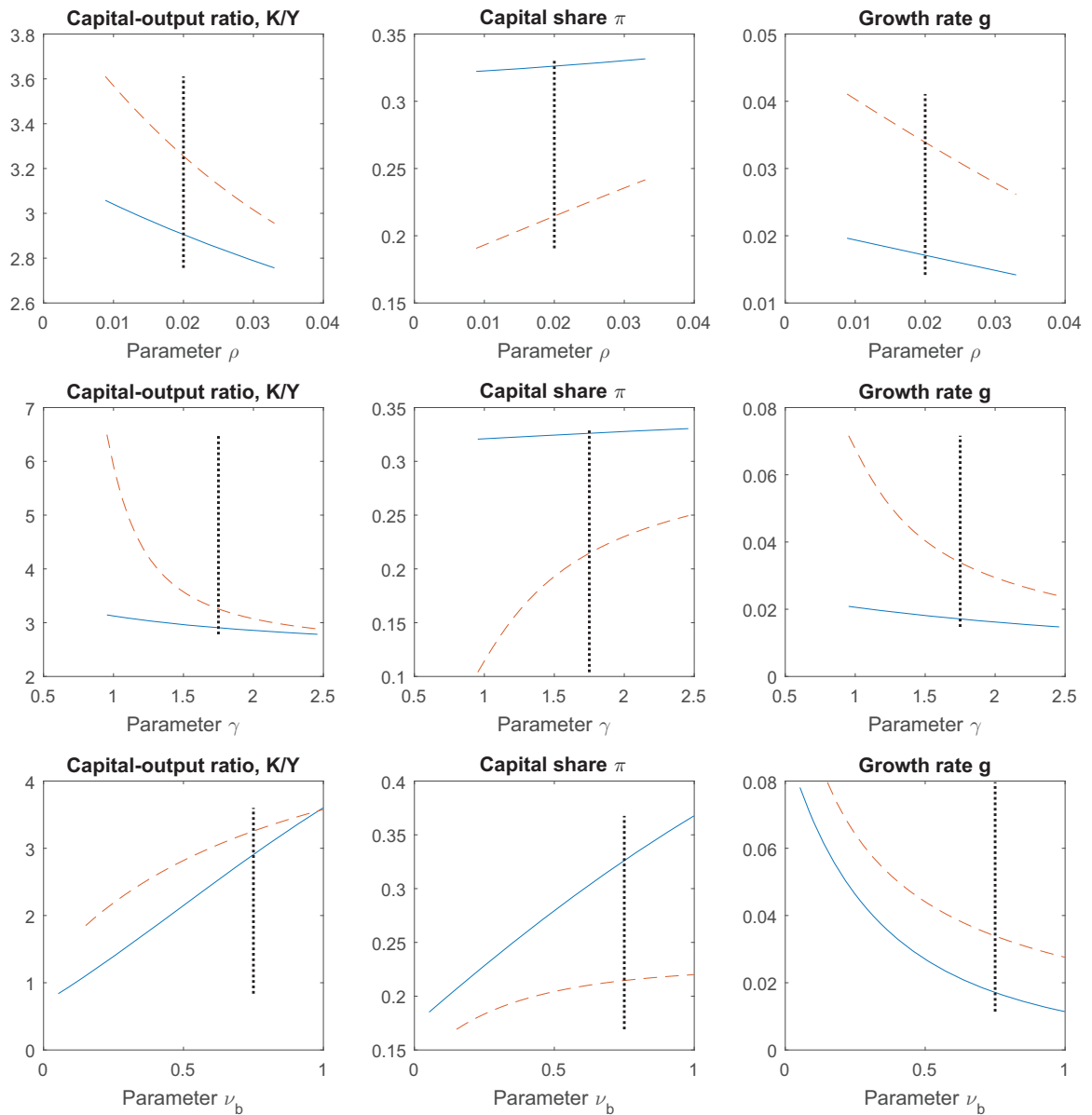
**Note:** Social planner allocation (dashed lines), decentralized equilibrium (solid lines). The vertical dotted line in each graph represents the baseline calibrated parameter value.

#### 4.5 Is the Decentralized Economy Characterized by Excessive Volatility?

In the data, we know that – irrespective of the concept utilized – labor shares are highly persistent and variable.<sup>23</sup> Although bounded within the unit interval and theoretically stationary, in the data labor income shares often appear to be characterized

<sup>23</sup>For international evidence see Jalava et al. (2006); Bengtsson (2014); Mućk et al. (2018).

**FIGURE 5: DEPENDENCE OF BGP ON  $\rho$ ,  $\gamma$  AND  $\nu_b$ , DA VS. SP.**



**Note:** Social planner allocation (dashed lines), decentralized equilibrium (solid lines). The vertical dotted line in each graph represents the baseline calibrated parameter value.

by marked volatility and long swings. In particular, around 80% of total labor share volatility in the US (1929–2015) has been due to fluctuations in medium-to-long run frequencies (beyond the 8 year mark). As opposed to the short-run component of the labor share, its medium-to-long run component has also been procyclical (Growiec et al., 2018).

Other than undermining the case for aggregate Cobb–Douglas production, this also raises the question of whether our framework can generate and rationalize these long cycles. Growiec et al. (2018) have confirmed this conjecture for the decentralized allocation of the current model. The question is however equally interesting for the social planner case. Are cycles in factor income shares socially optimal? If so, (stabilization) policies to mitigate labor share or real volatility might be appraised differently.<sup>24</sup>

**Table 3** makes the relevant comparisons across our maintained cases. It shows that the decentralized allocation features relatively shorter cycles but also faster convergence to the BGP. Hence, it cannot be claimed directly that the decentralized equilibrium has *excessive volatility* of the labor shares. If both allocations were to start from the same initial point outside of the BGP then the decentralized allocation would exhibit a greater frequency but smaller amplitude of cyclical variation.

**TABLE 3: DYNAMICS AROUND THE BGP**

Allocation	Baseline		C–D		Piketty	
	$\sigma = 0.7$		$\sigma = 1$		$\sigma = 1.25$	
	$\xi = -0.43$		$\xi = 0$		$\xi = 0.2$	
	DA	SP	DA	SP	DA	SP
Pace of convergence* (% per year)	6.3%	4.2%	5.8%	3.7%	5.2%	2.9%
Length of full cycle <sup>†</sup> (years)	52.6	76.7	79.8	83.2	144.0	100.3
Labor Share Cyclicalilty	+	+	0	0	–	–
Amplitude of $1 - \pi$ relative to $y/k$	62.0%	48.0%	N/A	N/A	28.0%	44.0%

**Note:** \* computed as  $1 - e^{rr}$  where  $rr < 0$  is the real part of the largest stable root; <sup>†</sup> computed as  $2\pi/ir$  where  $ir > 0$  is the imaginary part of two conjugate stable roots (if they exist). “N/A” denotes not available/applicable.

Having scrutinized the robustness of this dynamic result by extensively altering the model parametrization, we conclude that while the decentralized equilibrium generally exhibits shorter cycles, the ordering of both allocations in terms of the pace of convergence can sometimes be reversed. This finding lends partial support to the claim that the decentralized equilibrium is perhaps likely to feature greater labor share volatility compared to the social optimum. However, it is worthwhile to point out that oscillations in the labor income share can still be socially optimal in this model.

<sup>24</sup>By design, our analysis focuses only on endogenous long swings in factor shares. The deterministic character of the model precludes any conclusions regarding the magnitude and persistence of short-run fluctuations.

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Moreover, we also obtain quantitative predictions on the cyclical co-movement of the original model variables (including the economic growth rate  $g$  and the labor share  $1 - \pi$ ).<sup>25</sup> It turns out, both for the decentralized and optimal allocation, that all variables except for the consumption-capital ratio  $u = c/k$  oscillate when converging to the steady state, with the same frequency of oscillations. The level of capital augmenting technology  $\lambda_b$ , the “lab equipment” term  $x$ , and labor augmenting R&D employment  $\ell_a$  are always counter-cyclical, employment in production  $\ell_Y$  is always procyclical, whereas the cyclicity of capital augmenting R&D  $\ell_b$  is ambiguous (in the baseline calibration,  $\ell_b$  is pro-cyclical in the decentralized allocation but countercyclical in optimal one). Furthermore, as long as capital and labor are gross complements, the labor income share  $1 - \pi$  is unambiguously pro-cyclical as well. These features of cyclical co-movement align well with the empirical evidence for the US medium-term cycle. In particular, the US labor share is indeed procyclical over the medium-to-long run – despite its counter-cyclicity along the business cycle (Growiec et al., 2018; Mućk et al., 2018).

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<sup>25</sup>This is done by inspecting the eigenvector associated with the largest stable root of the Jacobian of the system at the steady state.

## 5 Conclusions

Endogenous growth theory tends to suggest that the socially optimal level of activity dominates the decentralized outcome. In this paper, we have confirmed this statement using a micro-founded, calibrated two-sector R&D endogenous growth model. Due to the existence of imperfect competition, externalities and duplication possibilities in production, the decentralized allocation produces a socially suboptimal level of R&D, and particularly too little labor augmenting R&D. This, in addition to a suboptimal level of capital accumulation, translates into too few efficient units of capital  $\lambda_b \cdot k/y$  at the BGP and thus too low equilibrium growth.

But what of the labor share? Despite its importance, the conclusions for this variable have not yet been drawn in the literature. Our objective was to bridge that knowledge gap. We found that if the elasticity of factor substitution is below unity (as the bulk of evidence suggests for the aggregate US economy), then the decentralized labor share is indeed socially suboptimal. The difference, moreover, is large, around 17%.

Effectively, the only parameter which reverses this ordering is the elasticity of substitution. However it tends to yield counter-factual outcomes otherwise. For example an elasticity of  $\sigma = 1.25$ , only slightly above Cobb Douglas, produces a decentralized labor share above the social planner one, but then the latter is as low as 0.52; as a simple point of comparison, according to the ILO definition of the labor share (using annual data from 1960 to the present), no G7 country has fallen below a labor share of 0.5. Moreover such a mild perturbation away from Cobb–Douglas already produces equilibrium per capital growth rates of around 6%.

In the future, our results should be contrasted with findings from a highly needed prospective study of optimal factor shares under inequality in factor ownership. Such a study could uncover the associated efficiency vs. inequality trade-off. We expect that the discrepancy between the optimal and decentralized labor share would then be even larger than 17% because the social planner would increase the labor share not just to improve efficiency of production under gross complementarity ( $\sigma < 1$ ) but also to reduce income inequality (given that capital incomes tend to be more concentrated).

## References

- Acemoglu, D. (2003). Labor-and Capital-Augmenting Technical Change. *Journal of the European Economic Association* 1, 1–37.
- Acemoglu, D. (2009). *Introduction to Modern Economic Growth*. Princeton University Press.
- Aghion, P., U. Akcigit, A. Bergeaud, R. Blundell, and D. Hémous (2015). Innovation and Top Income Inequality. Working paper 21247, National Bureau of Economic Research.
- Alvarez-Pelaez, M. J. and C. Groth (2005). Too little or too much R&D? *European Economic Review* 49(2), 437–456.
- Antràs, P. (2004). Is the U.S. Aggregate Production Function Cobb–Douglas? New Estimates of the Elasticity of Substitution. *Contributions to Macroeconomics* 4.
- Arrow, K., H. B. Chenery, B. S. Minhas, and R. M. Solow (1961). Capital-Labor Substitution and Economic Efficiency. *Review of Economics and Statistics* 43, 225–250.
- Atkinson, A., T. Piketty, and E. Saez (2011). Top incomes in the long run of history. *Journal of economic literature* 49, 3–71.
- Azariadis, C. (1996). The Economics of Poverty Traps - Part One: Complete Markets. *Journal of Economic Growth* 1, 449–486.
- Barro, R. J. and X. X. Sala-i-Martin (2003). *Economic Growth*. MIT Press.
- Bénassy, J.-P. (1998). Is there always too little research in endogenous growth with expanding product variety? *European Economic Review* 42(1), 61–69.
- Bengtsson, E. (2014). Labour's share in twentieth-century Sweden: a reinterpretation. *Scandinavian Economic History Review* 62(3), 290–314.
- Caselli, F. (2005). Accounting for Cross-Country Income Differences. In P. Aghion and S. Durlauf (Eds.), *Handbook of Economic Growth*. North-Holland.
- Chirinko, R. S. (2002). Corporate taxation, capital formation, and the substitution elasticity between labor and capital. *National Tax Journal* 60, 339–355.
- Chirinko, R. S. (2008).  $\sigma$ : The Long and Short of It. *Journal of Macroeconomics* 30, 671–686.

- Chirinko, R. S. and D. Mallick (2017). The Substitution Elasticity, Factor Shares, and the Low-Frequency Panel Model. *American Economic Journal: Macroeconomics* 9(4), 225–253.
- Cooley, T. F. (1995). *Frontiers of Business Cycle Research*. Princeton: Princeton University Press.
- de La Grandville, O. (1989). In Quest of the Slutsky Diamond. *American Economic Review* 79, 468–481.
- de La Grandville, O. (2012). How Much Should a Nation Save? A New Answer. *Studies in Nonlinear Dynamics and Econometrics* 16(2), 1–36.
- de La Grandville, O. (2016). *Economic Growth: A Unified Approach*. Second edition. Cambridge University Press.
- Dixit, A. K. and J. E. Stiglitz (1977). Monopolistic Competition and Optimum Product Diversity. *American Economic Review* 67, 297–308.
- Groth, C. and J. B. Madsen (2016). Medium-term Fluctuations and the “Great Ratios” of Economic Growth. *Journal of Macroeconomics* 49(C), 149–176.
- Growiec, J., P. McAdam, and J. Mućk (2018). Endogenous Labor Share Cycles: Theory and Evidence. *Journal of Economic Dynamics and Control* 87, 74–93.
- Growiec, J. and J. Mućk (2016). Isoelastic Elasticity of Substitution Production Functions. Working Papers 2016-001, Warsaw School of Economics, Collegium of Economic Analysis.
- Guo, J. T. and K. J. Lansing (2009). Capital-Labour Substitution and Equilibrium Indeterminacy. *Journal of Economic Dynamics and Control* 33(12), 1991–2000.
- Herrendorf, B., C. Herrington, and kos Valentinyi (2015). Sectoral Technology and Structural Transformation. *American Economic Journal: Macroeconomics* 7(4), 104–133.
- Hicks, J. (1932). *The Theory of Wages*. Macmillan London.
- Jaimovich, N. and H. E. Siu (2012). The Trend is the Cycle: Job Polarization and Jobless Recoveries. NBER Working Papers 18334, National Bureau of Economic Research, Inc.
- Jalava, J., M. Pohjola, A. Ripatti, and J. Vilmunen (2006). Biased Technical Change and Capital-Labour Substitution in Finland, 1902-2003. *The B.E. Journal of Macroeconomics* 6(1), 1–20.



- 
- Jones, C. I. (1995). R&D-Based Models of Economic Growth. *Journal of Political Economy* 103, 759–84.
- Jones, C. I. (1999). Growth: With or Without Scale Effects? *American Economic Review* 89(2), 139–144.
- Jones, C. I. (2005). Growth and Ideas. In P. Aghion and S. Durlauf (Eds.), *Handbook of Economic Growth*. North-Holland.
- Jones, C. I. and J. Kim (2018). A Schumpeterian Model of Top Income Inequality. *Journal of Political Economy*, forthcoming.
- Jones, C. I. and J. C. Williams (2000). Too Much of a Good Thing? The Economics of Investment in R&D. *Journal of Economic Growth* 5(1), 65–85.
- Jones, L. E. and R. E. Manuelli (1990). A Convex Model of Equilibrium Growth: Theory and Policy Implications. *Journal of Political Economy* 98, 1008–1038.
- Kaas, L. and L. von Thadden (2003). Unemployment, Factor Substitution, and Capital Formation. *German Economic Review* 4, 475–495.
- Karabarbounis, L. and B. Neiman (2014). The Global Decline of the Labor Share. *Quarterly Journal of Economics* 129(1), 61–103.
- Kennedy, C. (1964). Induced bias in innovation and the theory of distribution. *Economic Journal* 74, 541–547.
- Klump, R. (2002). Verteilung und Wirtschaftswachstum: Eine neoklassische Verallgemeinerung des Wachstumsmodell von Kaldor. In L. Menkhoff and F. Sell (Eds.), *Zur Theorie, Empirie und Politik der Einkommensverteilung*, pp. 11–26. Berlin-Heidelberg-New York.
- Klump, R. and O. de La Grandville (2000). Economic Growth and the Elasticity of Substitution: Two Theorems and Some Suggestions. *American Economic Review* 90, 282–291.
- Klump, R., P. McAdam, and A. Willman (2007). Factor Substitution and Factor Augmenting Technical Progress in the US. *Review of Economics and Statistics* 89, 183–192.
- Klump, R., P. McAdam, and A. Willman (2012). Normalization in CES Production Functions: Theory and Empirics. *Journal of Economic Surveys* 26, 769–799.
- Klump, R. and H. Preissler (2000). CES Production Functions and Economic Growth. *Scandinavian Journal of Economics* 102, 41–56.

- Kortum, S. (1993). Equilibrium R&D and the Patent-R&D Ratio: U.S. Evidence. *American Economic Review* 83(2), 450–457.
- León-Ledesma, M. A. and M. Satchi (2018). Appropriate Technology and Balanced Growth. *Review of Economic Studies*, forthcoming.
- Li, C.-W. (2000). Endogenous vs. Semi-endogenous Growth in a Two-R&D-Sector Model. *Economic Journal* 110(462), C109–22.
- Liesner, T. (1989). *One Hundred Years of Economic Statistics*. Facts on File Publications.
- McAdam, P. and A. Willman (2013). Medium run redux. *Macroeconomic Dynamics* 17, 695–727.
- Mućk, J. (2017). Elasticity of substitution between labor and capital: robust evidence from developed economies. NBP Working Papers 271, Narodowy Bank Polski, Economic Research Department.
- Mućk, J., P. McAdam, and J. Growiec (2018). Will the ‘True’ Labor Share Stand Up? An Applied Survey on Labor Share Measures. *Journal of Economic Surveys* 32, 961–984.
- Oberfield, E. and D. Raval (2018). Micro Data and Macro Technology. *Econometrica*, forthcoming.
- Palivos, T. and G. Karagiannis (2010). The Elasticity Of Substitution As An Engine Of Growth. *Macroeconomic Dynamics* 14(5), 617–628.
- Pessoa, A. (2005). Ideas’ driven growth: the OECD evidence. *Portuguese Economic Journal* 4(1), 46–67.
- Piketty, T. (2014). *Capital in the Twenty-First Century*. Harvard University Press. Translated from French by Arthur Goldhammer.
- Piketty, T. and G. Zucman (2014). Capital is Back: Wealth-Income Ratios in Rich Countries, 1700-2010. *Quarterly Journal of Economics* 129(3), 1255–1310.
- Pitchford, J. (1960). Growth and the elasticity of substitution. *Economic Record* 36, 491–504.
- Ramsey, F. P. (1928). A Mathematical Theory of Saving. *Economic Journal* 38, 543–559.
- Rognlie, M. (2015). Deciphering the Fall and Rise in the Net Capital Share: Accumulation or Scarcity? *Brookings Papers on Economic Activity* 46(1 (Spring)), 1–69.
- Romer, P. M. (1990). Endogenous Technological Change. *Journal of Political Economy* 98, S71–S102.

- 
- Rowthorn, R. (1999). Unemployment, wage bargaining and capital-labour substitution. *Cambridge Journal of Economics* 23(4), 413–425.
- Saam, M. (2008). Openness to trade as a determinant of the macroeconomic elasticity of substitution. *Journal of Macroeconomics* 30, 691–702.
- Solow, R. M. (1960). Investment and technical progress. In K. Arrow, S. Karlin, and P. Suppes (Eds.), *Mathematical Methods in the Social Sciences*, pp. 89–104. Stanford University Press.
- Uzawa, H. (1961). Neutral Inventions and the Stability of Growth Equilibrium. *Review of Economic Studies* 28, 117–124.
- Villacorta, L. (2018). Estimating country heterogeneity in capital-labor substitution using panel data. Available at <https://sites.google.com/site/luccianovillacorta/>.

## A Steady State of the Transformed System

### A.1 Social Planner Allocation

The steady state of the transformed dynamical system implied by the social planner solution satisfies:

$$g = \hat{\lambda}_a = \hat{k} = \hat{c} = \hat{y} = A(\lambda_b^*)^\phi (x^*)^{\eta_a} (\ell_a^*)^{\nu_a} \quad (\text{A.34})$$

$$\gamma g + \delta + \rho = z \left( \pi + \frac{1 - \pi}{\ell_Y} \left( \frac{\eta_a \ell_a}{\nu_a} + \frac{\eta_b \ell_b}{\nu_b} \right) \right) \quad (\text{A.35})$$

$$g = z - \zeta \frac{\dot{a}}{k} - u - (\delta + n) \quad (\text{A.36})$$

$$d = B \left( \lambda_b^{-\omega} x^{\eta_b} \ell_b^{\nu_b} \right) \quad (\text{A.37})$$

$$(1 - \gamma)g + n - \rho = d \left( \frac{\pi}{1 - \pi} \frac{\ell_Y \nu_b}{\ell_b} + (\phi + \eta_a) \frac{\nu_b \ell_a}{\nu_a \ell_b} - \omega + \eta_b \right) \quad (\text{A.38})$$

$$(1 - \gamma)g + n - \rho = -g \left( \frac{\ell_Y \nu_a}{\ell_a} - \eta_a - \eta_b \frac{\ell_b \nu_a}{\ell_a \nu_b} \right) \quad (\text{A.39})$$

$$\frac{\pi}{\pi_0} = \left( \frac{\lambda_b}{\lambda_{b0}} \right)^\xi \left( \frac{z}{z_0} \right)^{-\xi} \quad (\text{A.40})$$

$$\frac{z}{z_0} = \frac{\lambda_b}{\lambda_{b0}} \left( \pi_0 + (1 - \pi_0) \left( \frac{x_0 \ell_Y}{x \ell_{Y0}} \right)^\xi \right)^{1/\xi}. \quad (\text{A.41})$$

This non-linear system of equations is solved numerically, yielding the unique steady state of the de-trended system, and thus the unique BGP of the model in original variables. All further analysis of the social planner allocation is based on the (numerical) linearization of the 5-dimensional dynamical system of equations (22)–(24), (2) and (5), taking the BGP equality (21) as given.

### A.2 The Decentralized Allocation

As the case of the social planner, the Euler equations and dynamics of state variables are rewritten in terms of *stationary* variables. The steady state of the transformed

system satisfies:

$$g = \hat{\lambda}_a = \hat{k} = \hat{c} = \hat{y} = A(\lambda_b^*)^\phi (x^*)^{\eta_a} (\ell_a^*)^{\nu_a} \quad (\text{A.42})$$

$$\gamma g + \rho = r - \delta \quad (\text{A.43})$$

$$g = z - \zeta \frac{\dot{a}}{k} - u - (\delta + n) \quad (\text{A.44})$$

$$d = B \left( \lambda_b^{-\omega} x^{\eta_b} \ell_b^{\nu_b} \right) \quad (\text{A.45})$$

$$g \frac{\ell_Y}{\ell_a} = r - \delta \quad (\text{A.46})$$

$$g = r - \delta + d \left( 1 - \frac{\pi}{1 - \pi} \frac{\ell_Y}{\ell_b} \right) \quad (\text{A.47})$$

$$r = \varepsilon \pi z \quad (\text{A.48})$$

$$\frac{\pi}{\pi_0} = \left( \frac{\lambda_b}{\lambda_{b0}} \right)^\xi \left( \frac{z}{z_0} \right)^{-\xi} \quad (\text{A.49})$$

$$\frac{z}{z_0} = \frac{\lambda_b}{\lambda_{b0}} \left( \pi_0 + (1 - \pi_0) \left( \frac{x_0}{x} \frac{\ell_Y}{\ell_{Y0}} \right)^\xi \right)^{1/\xi}. \quad (\text{A.50})$$

This non-linear system of equations is solved numerically, yielding the unique steady state of the de-trended system, and thus the unique BGP of the model in original variables. All further analysis of the decentralized allocation is based on the (numerical) linearization of the 5-dimensional dynamical system of equations (22')–(24'), (2) and (5), taking the BGP equality (31) as given.

## B Additional Calibrated Parameters

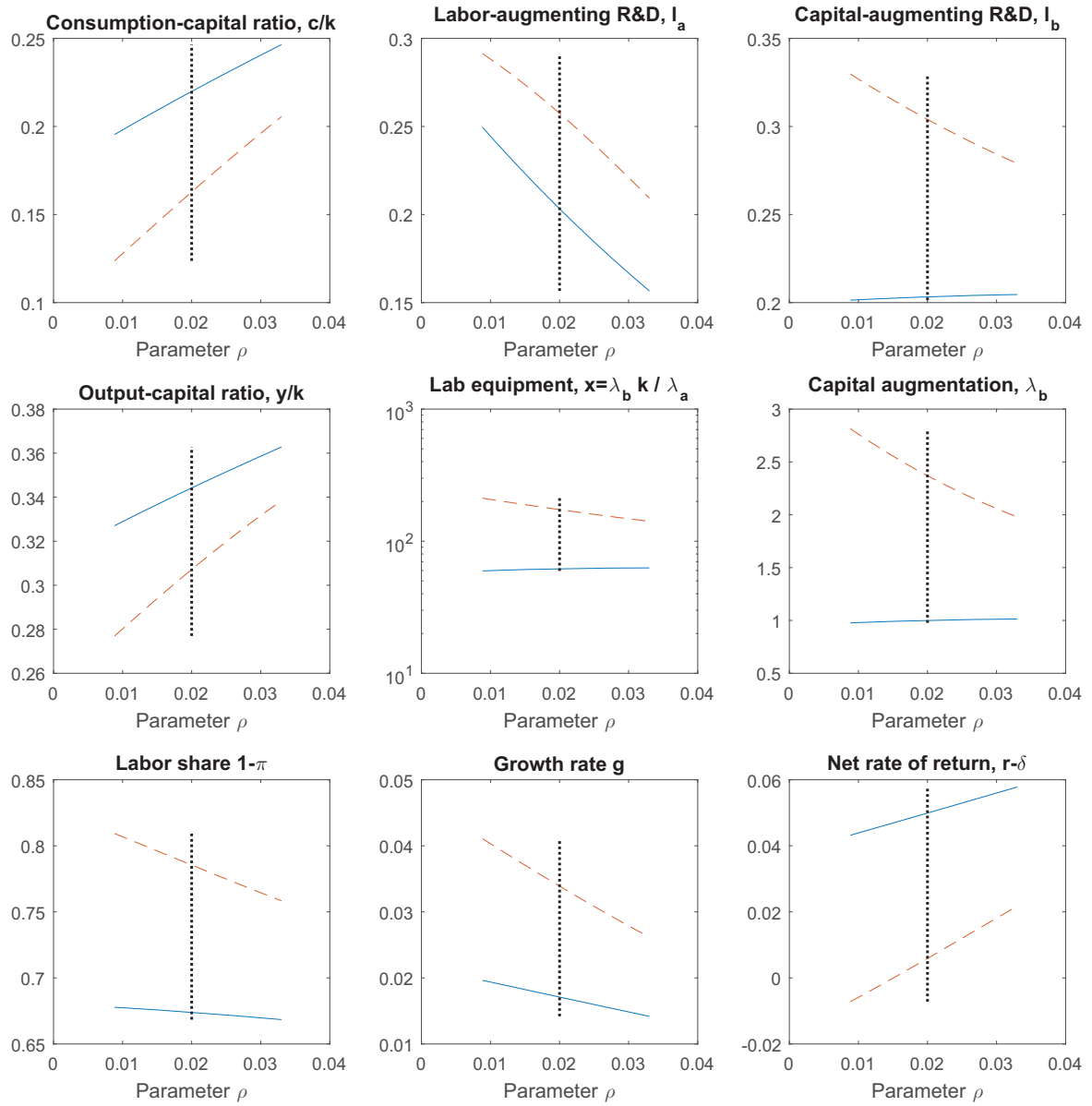
We solve the four remaining equations in system (A.42)–(A.50) with respect to the remaining parameters, see **Table B.1**. All these parameters are within admissible ranges. For instance, Pessoa (2005) estimates values for the obsolescence parameter between 0-15%; our endogenously determined value is thus centered in that range. Comparing  $\eta_a = 0.24$  with  $\eta_b = 0.13$  signifies that, first of all, lab equipment (effective capital augmentation of the R&D process) assuredly matters for R&D productivity, and second, that it is relatively more important for inventing new labor augmenting technologies than capital augmenting ones. Moreover, with  $\phi = 0.3$ , labor augmenting R&D – the ultimate engine of long-run growth – is substantially reinforced by spillovers coming from the capital augmenting R&D sector. On the other hand,  $\omega = 0.5$  means that the scope for capital augmenting R&D is quite strongly limited by decreasing returns. Given this benchmark calibration, as we said in the main text, the steady state is a saddle point.

**TABLE B.1: BASELINE CALIBRATION: ADDITIONAL PARAMETERS**

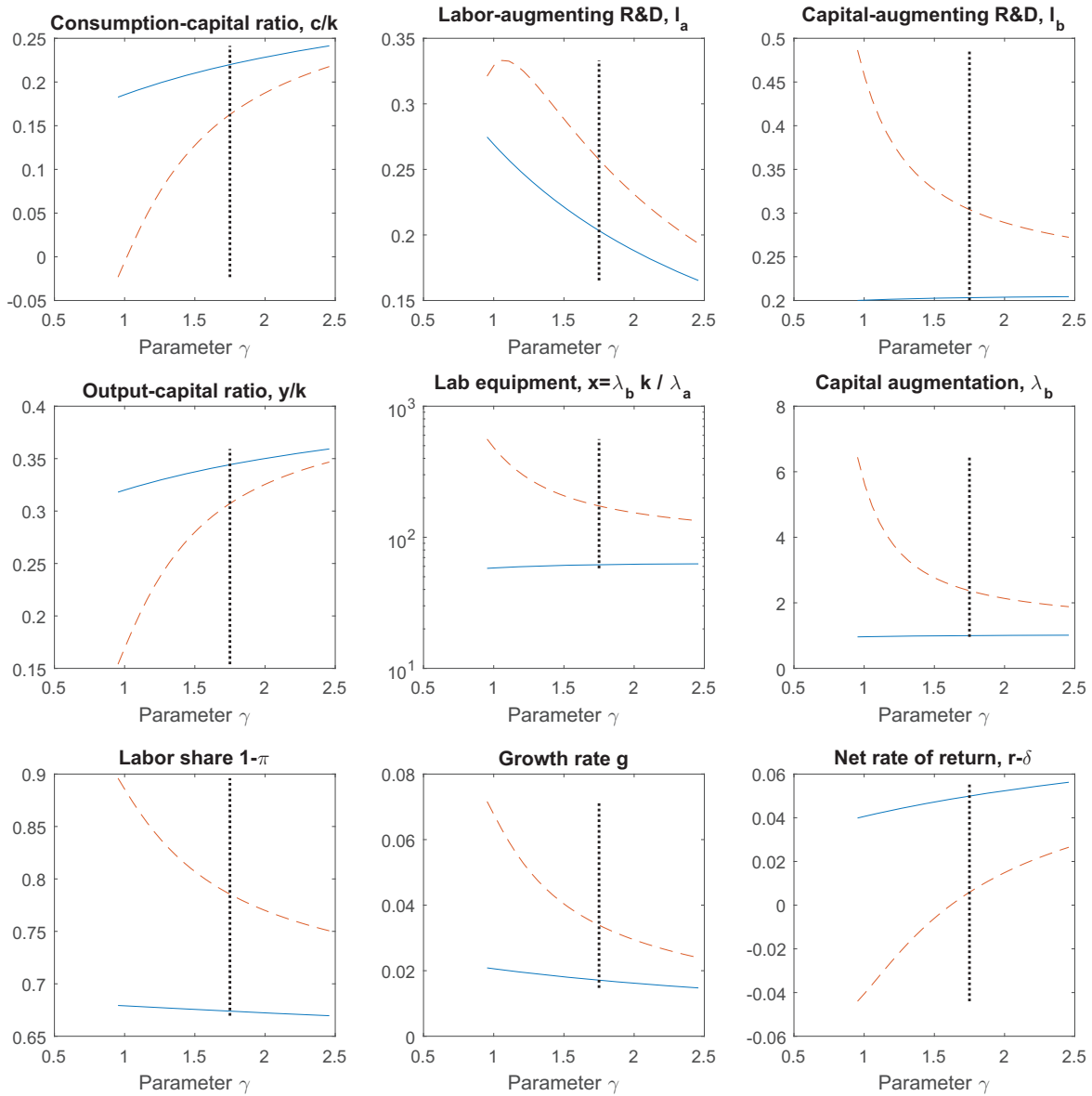
Parameter	Value	
<i>Labor augmenting R&amp;D</i>		
Unit productivity	$A$	0.02
Lab equipment exponent	$\eta_a$	0.24
<i>Capital Augmenting R&amp;D</i>		
Unit productivity	$B$	0.16
Lab equipment exponent	$\eta_b$	0.13
Degree of decreasing returns	$\omega$	0.50
Obsolescence rate	$d$	0.08
Spillover from capital to labor augmenting tech. change	$\phi$	0.30
Technology choice externality	$\zeta$	115.28

## C Additional Figures

**FIGURE C.1: COMPARING BALANCED GROWTH PATHS, DA VS. SP. DEPENDENCE ON THE TIME PREFERENCE.**

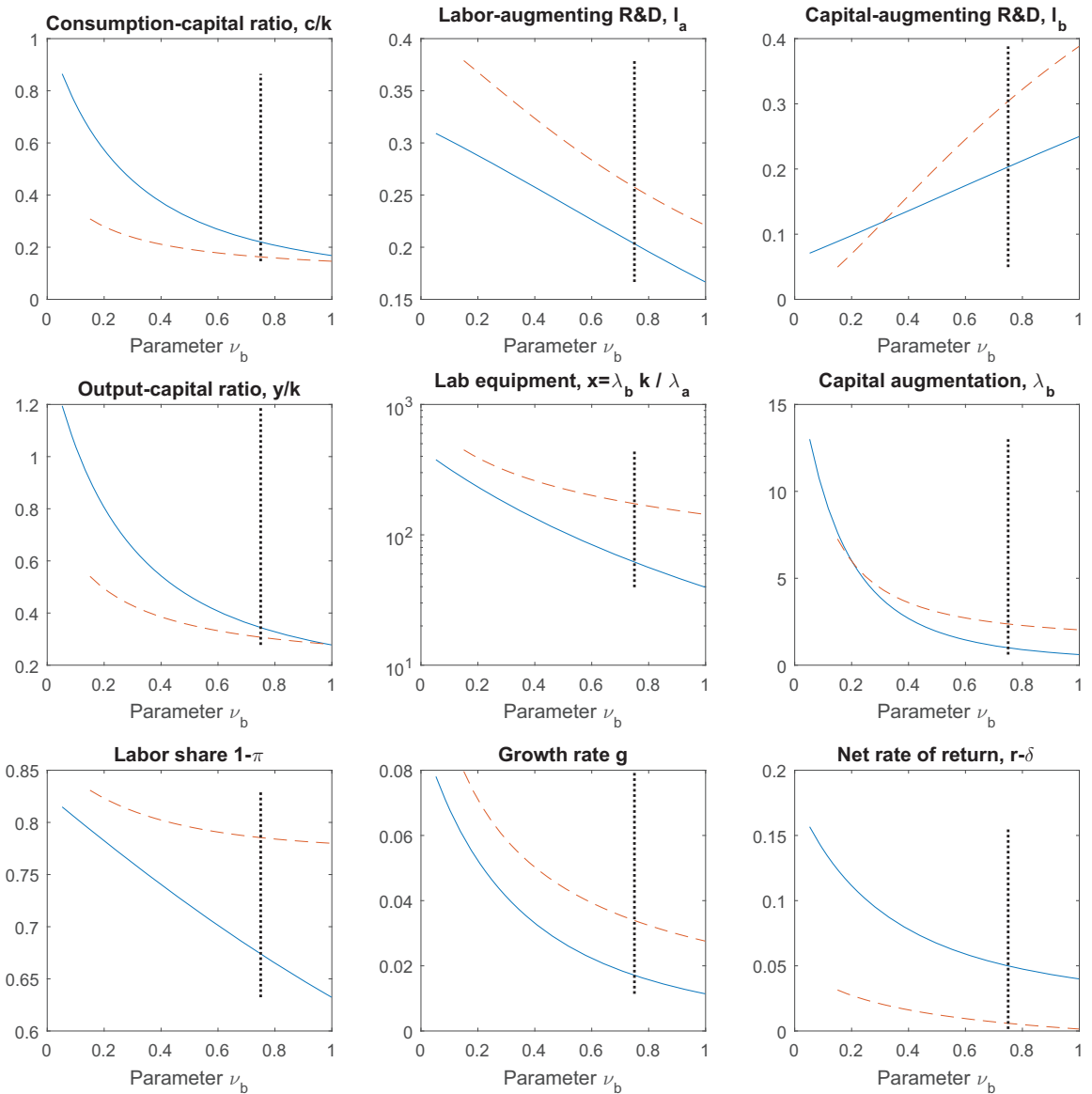


**FIGURE C.2: COMPARING BALANCED GROWTH PATHS, DA VS. SP.  
DEPENDENCE ON THE INTERTEMPORAL ELASTICITY OF SUBSTITUTION IN CONSUMPTION.**

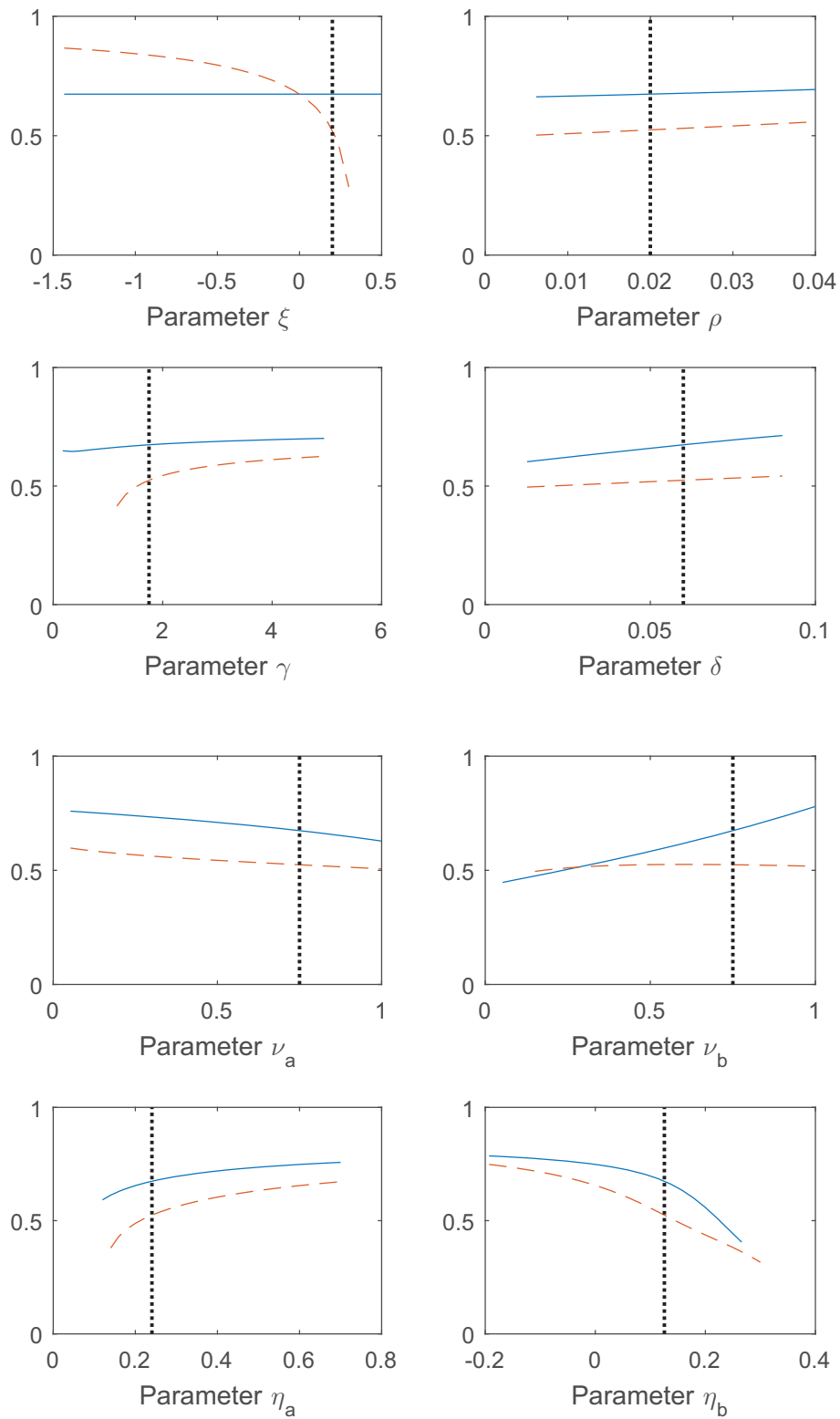




**FIGURE C.3: COMPARING BALANCED GROWTH PATHS, DA VS. SP. DEPENDENCE ON THE LAB EQUIPMENT EXPONENT IN CAPITAL AUGMENTING R&D.**



**FIGURE C.4: DEPENDENCE OF EQUILIBRIUM LABOR SHARE ON MODEL PARAMETERS, FOR THE ALTERNATIVE CALIBRATION OF  $\sigma = 1.25$  ( $\xi = 0.2$ ).**



**Notes:**  $1 - \pi$  on vertical axis; corresponding parameter support on the horizontal axis. Social planner allocation (dashed lines), decentralized equilibrium (solid lines). The vertical dotted line in each graph represents the baseline calibrated parameter value.

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