Endogenous Labor Income Cycles: Theory and Evidence*

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Abstract

Based on long US time series we document a range of empirical properties of the labor's share of GDP, including its substantial medium-run swings. We explore the extent to which these empirical regularities can be explained by a calibrated micro-founded long-run economic growth model with normalized CES technology and endogenous labor- and capital-augmenting technical change driven by purposeful directed R&D investments. It is found that dynamic macroeconomic trade-offs created by arrivals of both types of new technologies may lead to prolonged swings in the labor share due to oscillatory convergence to the balanced growth path as well as stable limit cycles via Hopf bifurcations. Both predictions are broadly in line with the empirical evidence.

Keywords: Labor income share, endogenous cycles, factor-augmenting endogenous technical change, R&D, CES.

JEL Codes: E25, E32, O33.

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1 Introduction

Labor's share of national income, once a focal point of debate from classical to post-war economists, has been rather neglected throughout the last half century. The recent surge of interest in this variable is due to two reasons. First, the mounting evidence that, rather than being approximately stable, the labor share is in fact quite volatile. Second, the observation that, in many countries, the labor share appears to have exhibited a secular decline since the 1970s.

Both phenomena have been studied in isolation and different factors have been used to explain each of them. The contribution of this article is to bridge these two hitherto disconnected literatures, and to contribute a new way of thinking about labor-share movements. To our knowledge, we are the first to document that the labor's share of GDP exhibits pronounced *medium-run swings*. On top of that, we are the first to assess the extent to which a calibrated, micro-founded endogenous growth model can shed light on these features.

Looking at the historical data[1], we document that there has been a hump-shaped pattern in the labor share, coupled with marked medium-term volatility and long persistence. Accordingly, we argue that the labor share should not necessarily be viewed as varying around a constant value, but rather around an endogenous medium-run trend which is itself variable. We also report its dynamic, medium-run correlation with other key macroeconomic variables. In total, these properties provide a motivation for setting up an endogenous, R&D-based growth model with endogenously-determined cycles in factor shares.

Existing models, it should be pointed out, are largely silent on these issues. Those with Cobb-Douglas production (unit elasticity, neutral technical change) imply constant factor shares. And business-cycle models with variable markups generate shares that fluctuate around a constant mean. Models endowed with a more general production specification (e.g., the neoclassical growth model with CES technology), on the other hand, do indicate a few critical tradeoffs; however, the profession has arguably not moved much beyond that.

Our framework suggests that conventional wisdom – i.e. that the labor share is largely driven by business cycles and is reasonably stable – may (in both respects) be misleading. First, we show that most of the variance of the labor share lies *beyond* business-cycle frequencies. Hence explaining labor-share movements with the aid of business-cycle mechanisms will only take us so far. Second, we show that, historically speaking, there is no compelling evidence that income shares are stable. Labor income shares may be better described as being determined by a long cycle[2]: starting from a low base in the late 1920s, stabilizing somewhat at a much higher value in the 1950-60s, and then declining from the 1970s onwards.

The challenge then is to model and rationalize that cycle. We do so through the lens of an endogenous growth model with non-neutral R&D-based technical change. Indeed,

[1] Annual data on the labor share of income exists from 1929, and quarterly series exists from 1947q1. We analyze the properties of both series in the empirical sections of this paper.

[2] For long-dated historical analysis of factor developments, value added factor shares and wealth distribution for several countries, see Piketty (2014).

starting from basics, we know that under CES production income shares can be associated to the capital-output ratio and factor-augmenting technical progress. Given the strong mean reversion displayed in the former ratio, the technology nexus seems a more promising starting point.[3]

Our framework is kept relatively simple and focused; for example, short-run frictions or appended shock processes (otherwise popular in the literature) are excluded.[4] Not only are these not well grounded in theory, they also pertain to business-cycle frequencies. Instead, focusing on R&D-led endogenous growth, we treat our model as a laboratory to assess mechanisms able to explain and rationalize observed labor-share swings over the medium and long run. Having calibrated the model on US data, we perform numerical exercises allowing us to confirm that the interplay between endogenous growth channels indeed leads to oscillatory convergence to the long-run growth path, and sometimes even to stable, self-sustaining (limit) cycles.[5] We do so for both the decentralized and social-planner allocation.

Hence, as opposed to earlier contributions which viewed the labor income share as either stationary or subject to a secular downward trend, in this paper we pursue the possibility that the observed swings in this variable are driven by low-frequency oscillations. In our model with gross complementarity between capital and labor in the aggregate CES production function, such oscillations appear endogenously as an outcome of the interplay between capital- and labor-augmenting R&D. If the agents are sufficiently patient (a low discount rate) and/or flexible in allocating consumption across time (a high elasticity of intertemporal substitution in consumption), the subsequent arrivals of both types of innovations can lead to limit cycle behavior via Hopf bifurcations. In such case, irrespective of the initial conditions, the economy converges to a stable cyclical path where all trendless macroeconomic variables (such as the labor share) oscillate indefinitely around the steady state. Such oscillations have a predetermined frequency and amplitude. Under our baseline calibration, however, the economy does not feature limit cycles, but exhibits oscillatory convergence to a balanced growth path (and a steady state in terms of trendless variables) instead. Along this convergence path, the labor income share and other trendless variables of the model are subject to dampened oscillations, with a predetermined oscillation frequency and an exponentially decreasing magnitude.

The structure of the paper is as follows. Section 2 documents the empirical evidence for medium-run swings in the labor share. We find, using a variety of tools, that the labor income share has a complicated makeup: it is highly persistent and volatile, appears to be

[3]Naturally, there may be other reasons behind the evolution of the labor share, such as changes in "institutions" and policy reforms, preferences, taxes and so on. But there seems to be limited consensus in the literature that, in a *medium-to-long term perspective*, that these variables have changed to such a degree and in such a commensurate manner as to be compatible with that labor-share pattern.

[4]For example, quadratic factor adjustment costs, habit formation in consumption, search-matching frictions, auto-regressive "shock" processes.

[5]Classic studies of economic cycles in this vein include Kaldor (1940), Goodwin (1951). Dockner (1985) and Feichtinger (1992) provides examples of and intuitions for limit cycles in economic systems. Faria and Andrade (1998) provide a simple limit-cycle model of aggregate investment.

characterized by breaks and has a frequency decomposition skewed to the medium and long run. Section 3 then briefly reviews the empirical literature describing the labor share, and the associated theoretical contributions.

Section 4 contains the setup and solution of the theoretical model. We consider a non-scale model of endogenous R&D-based growth with (a) two R&D sectors, giving rise to capital- as well as labor-augmenting innovations augmenting the "technology menu", (b) optimal factor-augmenting technology choice at the level of firms and (c) "normalized" local and global CES production functions. In addition, by assuming that new ideas follow Weibull (rather than, say, Pareto) distributions, aggregate CES production is retained in our framework.

Section 5 calibrates the model to US data and discusses the numerical results. We find that when compared with the socially optimal steady state, the decentralized labor share will in general be sub-optimal. We also investigate the effect perturbations in key parameters have on the labor share. We then formally consider the dynamic properties of the model in terms of oscillatory dynamics and stable limit cycles via Hopf bifurcations. Section 6 concludes.

2 Empirical evidence for medium-run swings in the labor share

In this section we explore the medium-term properties of the US labor share. First, the extent and importance of its medium-run swings is highlighted. Second, we formulate a range of associated *stylized facts*.

2.1 The historical time series of the US labor share

The historical annual time series of US labor's share of GDP, spanning 1929–2012, is presented in **Figure 1**.[6] Although we will also use quarterly series from 1947q1-2013q1. Following Gollin (2002), we adjust both series by proprietors' income (for details see Appendix A.1).

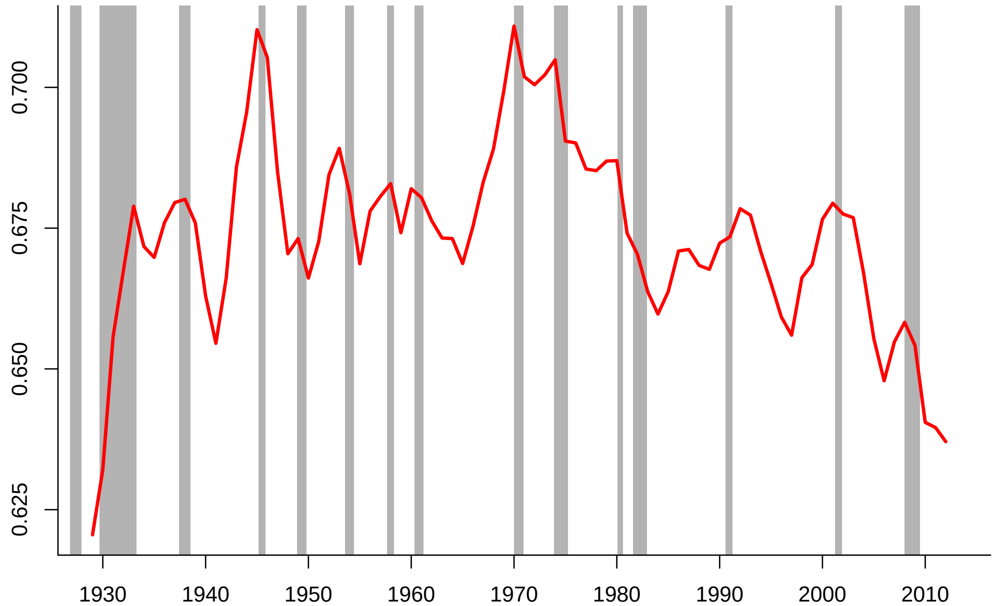
The labor share series as constructed here has all the properties identified in the earlier literature. First, it is counter-cyclical: it tends to rise during recessions (Young, 2004; Ríos-Rull and Santaeulàlia-Llopis, 2010).[7] Second, there indeed has been a marked decline since 1970s (Karabarbounis and Neiman, 2013). However, the benefit of using a long series is that we appreciate that this is only part of the story. Before the labor share began this decline, it showed an upward tendency. Indeed, when examined over the entire period, the historical series arguably looks part of a long cycle.[8]

[6] Quarterly series are available since 1947 and can be viewed in Figure B.1 in the Appendix.

[7] A regression of the quarterly labor income share on a constant and NBER recession dummy; a constant, an AR(1) terms and recession dummy; and a constant, an AR(1) term, a linear time trend and recession dummy, respectively yield the parameters on the dummy as $0.0061, 0.0044^{**}, 0.0037^{**}, 0.0040^{**}$. Superscript ** indicates statistical significance at 5%.

[8] Factor shares do not, to repeat the point made in the Introduction, appear to have inherited this type of pattern from the capital-output ratio, which is one of its fundamental determinants. See Figure B.2 in

Figure 1: The annual US labor share, 1929–2012



Note: Shaded areas represent recessions according to the NBER chronology at quarterly frequency. A table of summary statistics for the annual and quarterly labor income share is given in Appendix B.

This "hump-shaped" pattern has some interesting implications. According to Kaldor's (1961) stylized facts, factor shares should be broadly stable over time (or at least mean reverting). Notwithstanding that the labor income share is necessarily bounded, formal stationarity tests are inconclusive (see **Table B.2** in Appendix B). This may reflect the low power of tests in finite samples to distinguish between a non-stationary and near-stationary process, and/or the distorting presence of a structural break(s), Perron (1989). We will now examine these points in turn.

2.2 Persistence

Without loss of generality, assume that the labor share, y_t, follows an AR(1) process:

$$y_t = \mu + \rho_y y_{t-1} + \varepsilon_{y,t} \tag{1}$$

where the drift term μ captures the long-run mean, $\mu/(1-\rho_y)$ with $\rho_y \neq 1$, and $\varepsilon_{y,t} \sim \mathcal{N}(0, \sigma_y)$. Interest focus on both the value and stability of ρ_y. **Table 1** demonstrates that the labor share is a highly persistent, slowly-adjusting series (with ρ_y around 0.8 and 0.95 for annual and quarterly series respectively). This is robust to the inclusion of a linear or quadratic trend.[9]

appendix B.

[9]Although, naturally, these forms relax the assumption about the uniqueness of the labor share's equilibrium level.

Table 1: **AR(1) model estimates for the labor share**

	annual series			quarterly series		
	(1)	(2)	(3)	(1)	(2)	(3)
$\hat{\mu}$	0.108***	0.130***	0.176***	0.017	0.029**	0.053***
$\hat{\rho}_y$	0.840***	0.814***	0.739***	0.975***	0.958***	0.920***
$\hat{\beta}_1 \cdot 100$		−0.010**	0.020		−0.001	0.003**
$\hat{\beta}_2 \cdot 1000$			−0.004*			$-1.650e{-5}$***
$\hat{\sigma_y}$	0.008	0.008	0.007	0.004	0.004	0.004
R^2_{adj}	0.754	0.775	0.781	0.933	0.934	0.935
$\rho_y = 1$	[0.002]	[0.001]	[0.000]	[0.116]	[0.029]	[0.001]
	Quandt-Andrews unknown breakpoint test					
Max LR F-stat	[0.026]	[0.175]	[0.027]	[0.007]	[0.002]	[0.005]
Exp LR F-stat	[0.006]	[0.201]	[0.402]	[0.009]	[0.005]	[0.042]
Ave LR F-stat	[0.001]	[0.099]	[0.295]	[0.011]	[0.001]	[0.017]

Note: Asterisks ***, ** and * denote the rejection of null about parameter's insignificance at 1%, 5% and 10% significance level, respectively. Probability values in squared brackets. We employ a standard 15% sample trimming, and use probability values from Hansen (1997).
Specifications:
(1): $y_t = \mu + \rho_y y_{t-1} + \varepsilon_{y,t}$
(2): $y_t = \mu + \rho_y y_{t-1} + \beta_1 t + \varepsilon_{y,t}$
(3): $y_t = \mu + \rho_y y_{t-1} + \beta_1 t + \beta_2 t^2 + \varepsilon_{y,t}$

Further, in around 80% of the cases, there is evidence for one or more unknown structural parameter breakpoints using the Quandt-Andrews breakpoint tests. This in turn is validated by three further pieces of evidence: (a) rolling parameter estimates, (b) structural break tests, and (c) Markov switching. The latter can be seen as a indicator of persistence in the presence of complex, non-linear breaks.

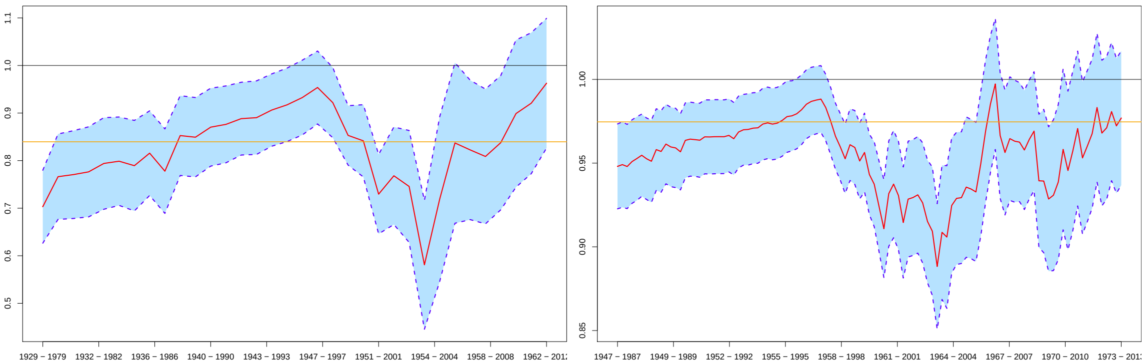
2.3 Rolling window estimation of the AR form

On point (a), **Figure 2** demonstrates that the auto-regression exhibits substantial variation between different sub-samples: from $0.61 - 0.96$ and from $0.88 - 1.00$ for annual and quarterly data, respectively.[10] The trajectory, though, is similar for both: U-shaped and with the lowest $\hat{\rho}_y$ occurring at more recent windows, e.g. around 2003. Apart from the adjustment parameters, the standard deviation of shocks driving the labor share $\hat{\sigma_y}$ also varies strongly and, independently of data frequency, is also characterized by a U-shaped trajectory. Here, the lowest $\hat{\sigma_y}$ is about 35% lower than the full sample estimates and is reached at the same window as in the $\hat{\rho}_y$ case.

[10] In order to check if the degree of persistence is roughly constant, the length of window is fixed at 50 and 40 years for the annual and quarterly series, respectively.

Figure 2: Rolling window estimates of ρ_y.

a: annual series (50Y) **b: quarterly series (40Y)**



Note: In addition to the point estimates plus confidence intervals, two reference lines are included at unity, and the sample mean of ρ_y.

2.4 Structural breaks

Regarding the dating of structural breaks, the results of the Bai and Perron (2003) procedure are reported in **Table 2**. The number of breaks is selected using the BIC criterion and is limited to at most 5. Irrespective of data frequency and the assumption on the DGP, two breakpoints are robustly identified: in the second half of the 1960s and in the first half of the 1980s. These dates delineate the successive periods characterized by an upward tendency, a short period of erratic fluctuations, and a downward tendency.

Table 2: Breaks in the labor share.

	annual series			quarterly series		
Specification	(1)	(2)	(3)	(1)	(2)	(3)
breaks	3	4	3	3	3	5
breakpoints	1942	1946	1942	1967Q4	1968Q1	1960Q3
	(1938,1953)	(1945,1956)	(1941,1943)	(1967Q1,1968Q2)	(1967Q3,1968Q2)	(1959Q1,1960Q2)
	1967	1968	1969	1980Q3	1986Q1	1969Q4
	(1963,1974)	(1967,1969)	(1968,1970)	(1980Q2,1981Q1)	(1985Q4,1986Q3)	(1969Q3,1970Q1)
	1980	1985	1994	2003Q2	1999Q4	1981Q2
	(1978,1981)	(1984,1987)	(1993,1995)	(2002Q1,2003Q4)	(1998Q4,2000Q1)	(1980Q4,1981Q4)
		1999				1992Q1
		(1995,2000)				(1991Q1,1992Q2)
						2001Q4
						(2001Q3,2002Q1)

Note: Specifications: (1) - only mean; (2) - linear trend; (3) - quadratic trend.

The evidence of at least two structural breaks indicates a nontrivial dynamic behavior of the labor share. In the next step, we run Markov-switching regressions to explore potential nonlinearities in the labor share's dynamic behavior.

2.5 Markov-Switching (MS) Results

Although MS models are typically applied in the business-cycle literature (Hamilton, 1989), e.g. to date recessions or identify asymmetrical business cycles, they might also be useful for capturing switches in lower frequency domains. While moving window estimation already allowed us to identify a U-shaped pattern in the series' changing persistence, Markov-switching autoregressive models are a valid extension of such analysis because here, possible changes in the variance are introduced directly into the full-sample estimation procedure. This method is therefore useful for verifying the existence of endogenous cycles driving the labor share.

Formally, the two-regimes Markov-switching AR(1) model is,

$$y_t = \mu_s + \rho_{y,s} y_{t-1} + \varepsilon_{y,t}, \qquad \varepsilon_{y,t} \sim \mathcal{N}(0, \sigma_{y,s}) \tag{2}$$

where $s_t = 1, 2$ is the latent variable indicating the current regime.

We consider four variants assuming regime-specific variance of shocks: with constant ρ_y and μ (specification no. 1), regime-specific μ (2), regime-specific ρ_y (3), and regime-specific μ and ρ_y (4).

Table 3: MS-AR(1) model estimates for the quarterly labor income share.

	raw series				de-trended series			
	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)
$\hat{\mu}$ or $\hat{\mu}_1$	0.017	0.020**	0.021	0.0328	−0.0001	−0.0007	−0.0001	−0.0007*
$\hat{\mu}_2$		0.021**		0.0066		0.0003		0.0004**
$\hat{\rho}_y$ or $\hat{\rho}_{y,1}$	0.975***	0.969***	0.968***	0.950***	0.920***	0.920***	0.721***	0.733***
$\hat{\rho}_{y,2}$			0.969***	0.991***			0.981***	0.983***
$\hat{\sigma}_{y,1}$	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005
$\hat{\sigma}_{y,2}$	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003
$\hat{\sigma}_{y,1}/\hat{\sigma}_{y,2}$	1.828	1.806	1.801	1.800	1.744	1.770	1.629	1.660
$\hat{p}_{1,1}$	0.968	0.965	0.965	0.966	0.951	0.951	0.978	0.977
$\hat{p}_{2,2}$	0.967	0.971	0.972	0.972	0.969	0.958	0.981	0.980
$\mathcal{D}_1$	7.812	7.143	7.143	7.353	5.102	5.102	11.364	10.870
$\mathcal{D}_2$	7.576	8.621	8.929	8.929	8.065	5.952	13.158	12.500
$n_1/(n_1+n_2)$	0.498	0.434	0.434	0.434	0.332	0.457	0.453	0.472

Note: Superscripts ***, ** and * denote significance at 1%, 5% and 10% significance level, respectively. $\mathcal{D}_i$, the (annual) duration of regime i, is calculated as $\frac{1}{4}(1-\hat{p}_{i,i})^{-1}$ and $n_1/(n_1+n_2)$ is a fraction of observations classified to the first regime.

The estimates are presented in **Table 3**, for both the raw and de-trended (quarterly) labor income share. It turns out that the differences in the error variance between regimes are robust to the model specification. The more volatile regime is characterized with a 75%

higher standard deviation of shocks. Once we relax the assumption of a constant adjustment parameter, we might identify two regimes: one with a lower variance of shocks and higher persistence, and a second one that is less persistent but more volatile.

Viewed from the medium-term perspective, the elements of the transition matrix are of interest. Note that $\hat{p}_{1,1}$ (i.e., the probability of staying in regime 1) and $\hat{p}_{2,2}$ are similar across all considered specifications. Besides, the small difference between $\hat{p}_{1,1}$ and $\hat{p}_{2,2}$ indicates that the expected duration does not vary between states so there is no strong evidence for asymmetric transition. The expected duration in each regime is around $8 - 12$ years (again lending weight to the medium-run interpretation of income shares).

Finally, using smoothed probabilities (see **Figure B.3** in the Appendix), we identify two periods with a relatively lower variance of shocks, and higher persistence: 1960-1975 and 1980-2000. These match very well the earlier identified breaks and common US-growth narratives (respectively, the high post-war growth and the 'Great Moderation").

2.6 Spectral analysis

The aforementioned empirical evidence of interesting nonlinear dynamics of the labor share is suggestive of the importance of the medium-run variation in this series. To verify the actual magnitude and frequency of medium-run swings, we turn to spectral techniques. Accordingly, the variation in the time series shall be split into three ranges in the frequency domain: high-frequency (with periodicity below 8 years), medium-frequency (periodicity between 8 and 50 years) and low-frequency oscillations (periodicity above 50 years). Note that the high-frequency component includes mainly business-cycle fluctuations (and noise). Furthermore, for spectral density estimation, the data should be demeaned or de-trended.[11] Having no priors, three variants of de-trending(demeaning) are considered.

Table 4 presents the estimated share of specific types of fluctuations in the total variance of the quarterly labor share series.[12] In the case of demeaned series, medium-frequency fluctuations are responsible for almost 40% of the whole volatility and only the cycles mapped into the low-frequency pass are (slightly) more important. As expected, de-trending the labor share series limits the contribution of low-frequency oscillations in the overall variance, and medium-term fluctuations become more important instead, with their share exceeding 50% and 70% for the series de-trended by a linear and quadratic trend, respectively.

To sum up, spectral analysis provides us with an additional argument for the importance of medium-run swings in the labor share. Irrespectively of the de-trending strategy, the share of medium-term fluctuations in the overall variance is meaningful and at least two times higher than the fraction linked to the short-run oscillations.

[11] The estimation on the spectral density is sensitive to the existence of both a unit root and a deterministic component in the series.

[12] We only use quarterly data (1949–2012) since spectral density estimation is highly sensitive to the number of observations.

Table 4: Share of specific frequencies in the observed variance (in %)

PERIODICITY (IN YEARS)	≥ 50	8-50	≤ 8
excluding the mean	42.8	37.9	19.3
excluding a linear trend	29.8	52.4	17.8
excluding a quadratic trend	0.4	72.6	27.0

Note: the shares are calculated with using periodogram estimates.

2.7 Stylized facts about the labor share's medium-term fluctuations

Thus the US labor share exhibits pronounced medium-term swings. To formulate a range of stylized facts about their properties, we follow Comin and Gertler (2006). Based on their definition, medium-term fluctuations are identified here as all cycles with periodicity between 8 and 50 years.[13]

The choice of method for extracting the medium-term component from the data is mostly determined by the frequency domain in question. Following earlier work on medium-term cycles (e.g., Comin and Gertler (2006), Correa-López and de Blas (2012)), we apply the Christiano and Fitzgerald (2003) (CF) approximation of the ideal band-pass filter.[14] The general strategy of isolating the medium-term component is the following. Because according to stationarity tests, both the labor share and virtually all its correlates in question are non-stationary, we transform our data into differences of log-levels and then apply the band-pass (CF) filter. Next, we cumulate the filtered data and demean. As a result, the extracted series represent percentage deviations from the long-run stochastic trend.[15]

The medium-term component extracted from the labor share is depicted in **Figure 3**. We see that the medium-frequency cyclical component is responsible for a significant part of the overall volatility of the series and that it has an important contribution to the scale of

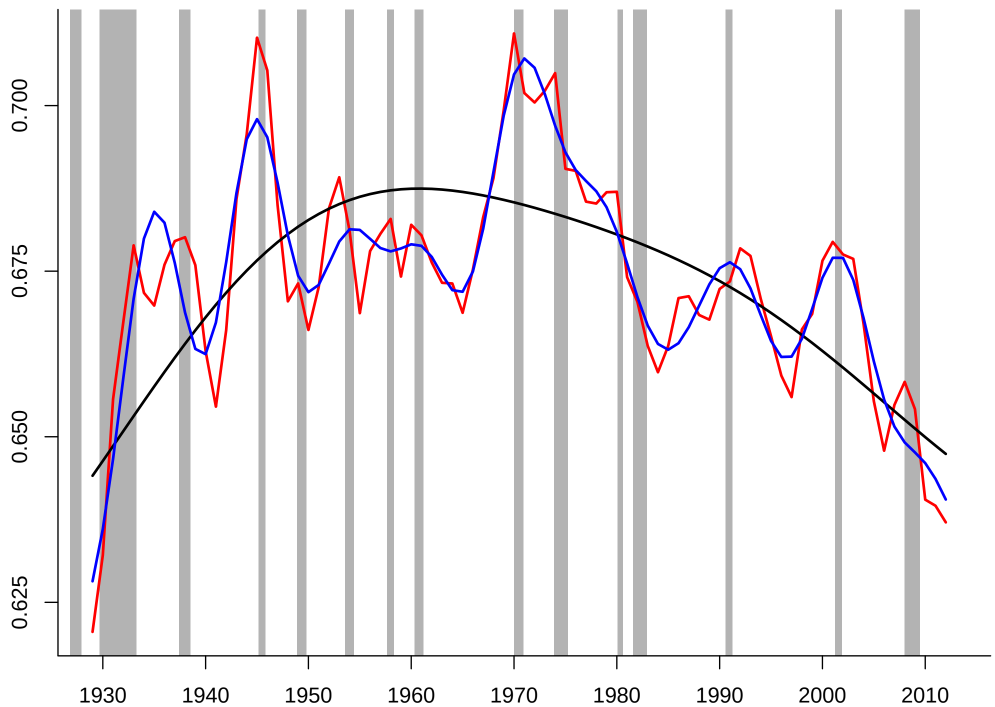
[13]More precisely, Comin and Gertler (2006) define medium-term business cycles as all fluctuations between 2 and 50 years, and then they divide such cycles into two components: the low-frequency component (with periodicity between 8 and 50 years) and the high-frequency component (with periodicity below 8 years). Note that the second one includes mainly the business-cycle fluctuations which are excluded from the current study.

[14]Being a low-pass filter, the Hodrick-Prescott filter, albeit arguably most popular in macroeconomics, cannot be used here. Alternatively, one could use the Baxter-King filter; there are two advantages of the CF procedure, however. First, applying the BK approximation incurs a loss in the number observations in the filtered series. Second, the fundamental assumption of the CF filter is the fact that data are generated by a random walk. Therefore, in our case the Christiano-Fitzgerald procedure is a more plausible way of extracting the low-frequency component.

[15]Interestingly, the results presented in this section are robust to the filtering strategy. For the labor share, general features are virtually the same when the CF procedure is applied to demeaned or de-trended data. For the remaining series, removing the linear trend leads to the similar conclusions.

deviation from long run trend at the turning points. Although isolating the medium- and high- frequency cycles reduces the volatility substantially, the remaining smoothed long-run trend still exhibits an hump shape (with a peak in the late 1950s).

Figure 3: The medium-term component of the annual US labor share



Note: the red, blue and black lines represent the raw series, the medium-term component and the long-run trend, respectively.

Akin to the real business cycle literature, we report the main features of the medium-term component of the labor share using moments of the filtered series: volatility (standard deviation), persistence (first-order autocorrelation) and co-movement (cross-correlation function) with selected other macroeconomic variables. Apart from the point estimates of the selected moments, we report also confidence intervals.[16] The general statistics are reported in **Table 5**.

[16]The confidence intervals have been computed using a bootstrap. More precisely, for a given moment we run an N sample with replacement. The CIs are the 0.025 and 0.975 quantiles of the obtained simulated distributions. In the table we use 5000 replications; a larger number did not change the results.

Table 5: Features of Labor Share's Medium-Term Component

	σ_{LS_t}	$\sigma_{LS_t}/\sigma_{GDP_t}$	$\rho_{LS_t,LS_{t-1}}$	ρ_{LS_t,GDP_t}
ANNUAL SERIES	1.450	0.290	0.940	0.480
	(1.230,1.640)		(0.900,0.970)	(0.270,0.650)
QUARTERLY SERIES	1.450	0.300	0.996	0.540
	(1.330,1.540)		(0.995,0.996)	(0.470,0.620)

Note: σ_{LS_t} and $\sigma_{LS_t}/\sigma_{GDP_t}$ denotes volatility in absolute term (percentage deviation from the long-run trend) and relative term (as a ratio to the GDP's volatility). $\rho_{LS_t,LS_{t-1}}$ and ρ_{LS_t,GDP_t} stand for the first-order autocorrelation and contemporaneous co-movement with product, respectively.

Based on this, we formulate following *stylized facts*:

(a) The medium-term component of the labor share is highly persistent.

(b) The labor share fluctuates substantially in the medium-term frequencies. The volatility of labor share is about 30% of the output volatility.

(c) The labor share is positively correlated with output. Even if the estimates of contemporaneous co-movement are not so high, they remain in stark contrast to the negative correlation repeatedly reported for the short-run component.

The pro-cyclicality of the medium-term component of the labor share is probably the most surprising *stylized fact*. Moreover, the phase shift between cyclical components of those variables suggests that highest correlation is observed if the labor share lagged by two years (though the difference from the contemporaneous case is not significant).

A wider range of statistics describing the labor share's co-movement with main macroeconomic variables is presented in **Table 6**. Virtually all pro-cyclical variables in the medium-term business cycles, for instance I_t, C_t, $hours_t$ and E_t, are also positively correlated with the labor share.

Our key observations here are as follows. First, although the contemporaneous co-movement of the labor share with R&D expenditures is not statistically significant, it becomes negative and significant once we lag the labor share by 3-4 years. This indicates that bursts in R&D activity can be viewed as a leading indicator for downward swings in the labor share 3-4 years later. A similar pattern is identified for the consumption-capital ratio: periods when consumption is relatively high are followed (in around 7 years) by similar downward swings. On the other hand, the opposite is found for the capital-output ratio: periods of capital-intensive production are followed (in around 7 years) by periods where the labor share rises.

Finally, on the skill-premium series, in the medium-term frequency range, the cyclical component of w_t^S/w_t^U is quite volatile and persistent. Furthermore, the contemporaneous co-movement with output is strongly negative. Interestingly, the pairwise correlation with the pro-cyclical labor share is just at the border of statistical significance.

Table 6: Selected Labor Share Cross Correlations

	Correlation with the Labor Share					Medium-Term Characteristics			
	ρ_{LS_t,x_t}	ρ_{LS_{t+k},x_t}	k	ρ_{LS_{t+k},x_t}	k	σ_{x_t}	$\sigma_{x_t}/\sigma_{y_t}$	$\rho_{x_t,x_{t-1}}$	ρ_{y_t,x_t}
$K_t^{private}/Y_t$	-0.18	0.75	7	-0.85	-5	6.17	1.24	0.96	-0.03
	(-0.38,0.04)	(0.64,0.83)		(-0.91,-0.77)		(5.39,6.84)		(-0.32,0.27)	(-0.32,0.27)
C_t	0.53	0.57	-1	-0.32	-10	3.44	0.69	0.96	0.87
	(0.37,0.68)	(0.41,0.69)		(-0.57,-0.05)		(2.97,3.91)		(0.94,0.97)	(0.8,0.92)
I_t	0.46	0.59	-3	-0.42	-10	11.83	2.39	0.94	0.51
	(0.27,0.64)	(0.46,0.7)		(-0.61,-0.18)		(9.94,13.55)		(0.91,0.97)	(0.34,0.66)
G_t	0.30	0.54	-3	-0.29	5	7.40	2.49	0.86	-0.02
	(0.09,0.49)	(0.29,0.73)		(-0.47,-0.06)		(5.96,8.68)		(0.77,0.92)	(-0.23,0.2)
C_t/K_t	0.15	0.78	-6	-0.77	7	6.15	1.24	0.97	0.14
	(-0.07,0.35)	(0.69,0.86)		(-0.85,-0.66)		(5.26,6.93)		(0.96,0.98)	(-0.14,0.39)
C_t/Y_t	-0.02	0.25	-10	-0.41	7	1.61	0.32	0.90	0.26
	(-0.22,0.18)	(-0.01,0.46)		(-0.59,-0.19)		(1.26,1.99)		(0.82,0.95)	(0.07,0.49)
I_t/Y_t	0.11	0.43	5	-0.55	-10	10.6	2.14	0.94	0.21
	(-0.05,0.28)	(0.22,0.61)		(-0.68,-0.36)		(9.09,11.92)		(0.91,0.96)	(-0.04,0.43)
$Employment_t$	0.42	0.62	-2	-0.47	-9	2.15	0.43	0.88	0.48
	(0.26,0.56)	(0.46,0.75)		(-0.65,-0.23)		(1.86,2.41)		(0.82,0.93)	(0.28,0.63)
$hours_t$	0.34	0.80	-3	-0.27	10	2.62	0.53	0.9	0.52
	(0.15,0.51)	(0.7,0.87)		(-0.54,0.03)		(2.22,2.98)		(0.85,0.94)	(0.33,0.68)
LP_t	0.42	0.49	2	-0.25	-10	3.97	0.8	0.98	0.83
	(0.23,0.58)	(0.27,0.67)		(-0.51,0.02)		(3.59,4.34)		(0.97,0.99)	(0.75,0.89)
RD_t/GDP_t	-0.3	0.73	-10	-0.62	3	12.29	2.48	0.96	0.13
	(-0.47,-0.12)	(0.59,0.84)		(-0.73,-0.5)		(10.15,14.34)		(0.93,0.98)	(-0.05,0.3)
RD_t	-0.07	0.59	-9	-0.45	4	13.85	2.79	0.97	0.42
	(-0.28,0.15)	(0.38,0.76)		(-0.62,-0.26)		(11.54,16.05)		(0.94,0.98)	(0.23,0.59)

Note: ρ_{y_t,x_t} and ρ_{LS_t,x_t} denote the contemporaneous cross-correlation for series x_t with output and the labor share. ρ_{LS_{t+k},x_t} reflects to the correlation of variable x_t with labor share lagged by k period. For the labor share the highest and the lowest cross-correlation with each series are reported. $\rho_{x_t,x_{t-1}}$ and σ_{x_t} denote the first-order autocorrelation and standard deviation from the long-run trend, respectively.

To conclude, the empirical evidence on the dynamic behavior of the medium-term component of the historical time series of the US labor share points out at its high persistence, substantial volatility, and interesting patterns of correlation with other macroeconomic variables. In the following section, we take a brief look at the literature on describing movements in the labor income share.

3 Associated literature

The literature describing movements in the labor share is diverse, encompassing different theoretical and empirical perspectives. A full review is beyond our purpose, but we can touch on some themes which are popular, and closely-related.

Naturally, the recent secular decline in labor income share poses a particular challenge to theory. The usual macroeconomic paradigm of Cobb-Douglas production coupled with isoelastic demand (leading to constant markups) leaves no room for the prolonged swings in factor shares observed in the data. The literature therefore tries to explain this phenomenon as departures from that benchmark.

For instance, movements in labor share have often been seen as reflecting transitory capital augmentation acting alongside the more standard labor-augmenting type of technical change (Acemoglu, 2003; Bentolila and Saint-Paul, 2003; Jones, 2005b; Klump, McAdam, and Willman, 2007). Given that capital and labor are gross complements, this pattern ensures asymptotically stable income shares while allowing fluctuations in the short-to-medium run.

At a less aggregate level, there is also the role of skill-biased technical change (SBTC). This is defined as a change in the production technology that favors skilled over unskilled labor by increasing the former's relative productivity and, therefore, relative demand. SBTC may operate through different channels (see Krusell, Ohanian, Ríos-Rull, and Violante, 2000; Arpaia, Pérez, and Pichelmann, 2009; Autor, Katz, and Kearney, 2008). Such developments will necessarily impact income shares both between labor types and between labor and capital.

At an even more disaggregate level, structural transformation within the economy may also help explain long-run trends in factor income shares, see Kongsamut, Rebelo, and Xie (2001), de Serres, Scarpetta, and de la Maisonneuve (2002), Ngai and Pissarides (2007), Acemoglu and Guerrieri (2008), Buera and Kaboski (2012). Think for example of the rise in importance of Manufacturing and Services and the decline of Agriculture. Such shifts may, for instance, impact income shares depending on the skills and bargaining power of the affected labor; the overall trend in mark-ups following from the sectoral changes (McAdam and Willman, 2004); firm size and age (Kyyrä and Maliranta, 2008), and so on.

These explanations all have attractions and weaknesses of one sort or another. Those based on factor-augmenting technical change, sectoral R&D and endogenous growth are ones that we are drawn to, largely for their simplicity and intuition. Accordingly, in the following sections we outline the model to be used.

4 Model

We now consider a non-scale model of fully endogenous R&D-based growth with (a) two R&D sectors, giving rise to capital- as well as labor-augmenting innovations augmenting the technology menu (Acemoglu, 2003), (b) optimal factor-augmenting technology choice at the level of firms, following Jones (2005b), and (c) normalized local and global CES production functions (Klump and de La Grandville, 2000). Before that, we put our contribution in the context of the existing literature.

4.1 Endogenous technology choice in the literature

Our starting point is Jones (2005b) who, building on Houthakker (1955), argued that the aggregate or "global" production function (GPF) can be viewed as the upper envelope (convex hull) of "local" production functions (LPF). The latter should exhibit low factor substitution reflecting the quasi fixity of ideas.[17] However, assuming that new techniques are independently and identically drawn from a Pareto distribution, Jones (2005b) demonstrates that the GPF actually tends to Cobb-Douglas (with technical change asymptotically labor-augmenting). The Pareto form, moreover, is appealing since, given its heavy-tailed distribution, it matches many economic phenomena (e.g., firm and city size, stock returns, income distribution). Plus it embodies a proportionality factor, i.e., $\mathbb{E}[x|x \geq \tilde{x}] \propto \tilde{x}$ which is intuitive in the modelling of ideas.

Notwithstanding, there is no *overwhelming* reason to choose this particular distribution, or to discard equally plausible alternatives. Indeed in our context, the framework is not without its drawbacks. First, a unitary elasticity of substitution is counter-factual (Klump, McAdam, and Willman, 2012). Likewise, whilst necessarily bounded, the labor share has been shown to fluctuate substantially with very long swings. Whilst aggregation towards the global Cobb-Douglas form does not preclude such fluctuations, it still leaves a gap in our knowledge of how factor income shares behave in the transition. Is that dynamic monotonic, oscillatory, self sustaining, or even unique? If oscillatory, would a suitable growth model generate cycles of plausible length?

Growiec (2008a,b) demonstrated that if new techniques are instead independently Weibull-distributed then the GPF is not Cobb-Douglas but CES. Furthermore, assuming that factor-augmenting technologies are inherently complex and consist of a large number of complementary components, the Weibull distribution should approximate the true productivity distribution better than anything else, including the Pareto (Growiec, 2013).

The argument is based on the extreme value property of the Weibull (de Haan and Ferreira, 2006): if one takes the minimum of n independent draws from some (sufficiently well-behaved) distribution, then as $n \to \infty$, this minimum will converge in distribution to the standard Weibull with its "shape" parameter, α, dependent on the shape of the underlying sampling distribution.

Taking the minimum corresponds to the case of complex technologies consisting of complementary components (e.g. Kremer, 1993) whose productivity is determined by that of their "weakest link". The same complementarity requirement, coupled with the assumption of limited substitution possibilities along the local production, implies also that capital and labor should be gross complements along the global CES production function. Accordingly, we can maintain the more empirically relevant CES global function.

[17] Often the LPF elasticity is motivated as Leontief. But perfect complementarity is a strong assumption with the counter-factual implication that shares of capital and labor in output approach one-half (de La Grandville, 2009). Furthermore, in the technology and growth literature, the Leontief form is usually ruled out given its dis-equilibrium implications for growth and optimal savings (Barro and Sala-i-Martin, 2003).

4.2 Local and global technology

Assume that the local production function takes the normalized CES or normalized Leontief form (de La Grandville, 1989; Klump and de La Grandville, 2000):

$$Y = Y_0 \left(\pi_0 \left(\frac{bK}{b_0 K_0} \right)^{\theta} + (1-\pi_0) \left(\frac{aL_Y}{a_0 L_{Y0}} \right)^{\theta} \right)^{\frac{1}{\theta}}, \tag{3}$$

In the normalization procedure of the CES LPFs, benchmark values have been assigned not only to output, capital and labor (Y_0, K_0, L_0), but also to the benchmark technology (b_0, a_0). In the following derivations, this benchmark technology will be identified with the *optimal* technology at time t_0.

Capital and labor are assumed to be gross complements along the LPF (i.e., $\theta < 0 \Leftrightarrow \sigma_{LPF} < 1$).[18] Under the assumption that the representative firm operates in a perfectly competitive environment, which is possible given constant returns, the capital share equals

$$\pi = \frac{\pi_0 \left(\frac{bK}{b_0 K_0} \right)^{\theta}}{\pi_0 \left(\frac{bK}{b_0 K_0} \right)^{\theta} + (1-\pi_0) \left(\frac{aL_Y}{a_0 L_{Y0}} \right)^{\theta}} = \pi_0 \left(\frac{b}{b_0} \right)^{\theta} \left(\frac{KY_0}{K_0 Y} \right)^{\theta}. \tag{4}$$

and the labor share amounts to $1-\pi$.[19] Thus the capital share is determined by capital augmenting technology and capital productivity.

From an economic point of view, the assumption that there is limited substitutability along the LPF is consistent with the "recipe" interpretation of particular production techniques (where the LPF is viewed as a list of instructions on how to transform inputs into output, that must be followed as closely as possible, cf. Jones (2005b)). It is thus clearly preferred over the alternative cases of $\theta \to 0$ (Cobb–Douglas LPFs) or $\theta \in (0,1)$ (gross factor substitutability along the LPF). These cases being respectively counter-factual and theoretically anomalous.[20,21]

The "technology menu" specified in the (a, b) space is given by

$$H(a,b) = \left(\frac{a}{\lambda_a} \right)^{\alpha} + \left(\frac{b}{\lambda_b} \right)^{\alpha} = N, \qquad \lambda_a, \lambda_b, \alpha, N > 0. \tag{5}$$

[18] Since all results derived in this paper also go through in the limiting case of Leontief LPFs, the CES specification of the LPF is not strictly necessary for the aggregate CES production function to obtain.

[19] We confine ourselves to constant-returns production functions, consistent with much of the aggregate evidence, e.g., Basu and Fernald (1997).

[20] Chirinko (2008) and Klump, McAdam, and Willman (2012) tabulate a number of empirical production studies over many different historical periods and countries, and find little evidence for unitary or above unitary forms. The most famous Cobb-Douglas finding was due toBerndt (1976), although it should be recalled that his result concerned only the US manufacturing sector.

[21] Solow (1956) showed in the neoclassical growth model that a CES function with an elasticity of substitution greater than one can generate sustained growth (even without technical progress). de La Grandville (2009) discusses the precise threshold conditions associated to this perpetual-growth scenario.

The technology menu is thus downward sloping in (a, b), capturing the trade-off between the available unit factor productivities (UFPs) of capital and labor. The technology menu can be understood as a *contour line of the cumulative distribution function* of the joint bivariate distribution of capital- and labor-augmenting ideas ($\tilde{b}$ and $\tilde{a}$, respectively). Under independence of both dimensions (so that marginal distributions of $\tilde{b}$ and $\tilde{a}$ are simply multiplied by one another), equation (5) is obtained iff the marginal distributions are Weibull with shape parameter $\alpha > 0$:

$$P(\tilde{a} > a) = e^{-\left(\frac{a}{\lambda_a}\right)^\alpha}, \qquad P(\tilde{b} > b) = e^{-\left(\frac{b}{\lambda_b}\right)^\alpha}, \tag{6}$$

for $a, b > 0$. Under such parametrization, we have $P(\tilde{a} > a, \tilde{b} > b) = e^{-\left(\frac{a}{\lambda_a}\right)^\alpha - \left(\frac{b}{\lambda_b}\right)^\alpha}$, and thus $N = -\ln P(\tilde{a} > a, \tilde{b} > b) > 0$. We assume N to be constant, and for λ_a and λ_b to grow as an outcome of factor-augmenting R&D (Growiec, 2013).

Firms then choose the technology pair (a, b) optimally, subject to the current technology menu, such that their profit is maximized:[22,23]

$$\max_{a,b} \left\{ Y_0 \left(\pi_0 \left(\frac{bK}{b_0 K_0} \right)^\theta + (1 - \pi_0) \left(\frac{a L_Y}{a_0 L_{Y0}} \right)^\theta \right)^{\frac{1}{\theta}} \right\} \qquad s.t. \qquad \left(\frac{a}{\lambda_a} \right)^\alpha + \left(\frac{b}{\lambda_b} \right)^\alpha = N. \tag{7}$$

It is easily verified that at time $t = t_0$ (the point of normalization), the optimal choice is then,

$$a_0^* = (N(1 - \pi_0))^{\frac{1}{\alpha}} \lambda_{a0}, \qquad b_0^* = (N\pi_0)^{\frac{1}{\alpha}} \lambda_{b0}, \tag{8}$$

The above values of a_0^* and b_0^* will be used as a_0 and b_0 in the normalization at the local level in all subsequent derivations. For any other $t \neq t_0$, the optimal technology choices are:

$$\left(\frac{a}{a_0} \right)^* = \frac{\lambda_a}{\lambda_{a0}} \left(\pi_0 \left(\frac{\lambda_b}{\lambda_a} \frac{\lambda_{a0}}{\lambda_{b0}} \frac{K L_{Y0}}{L_Y K_0} \right)^{\frac{\alpha\theta}{\alpha - \theta}} + 1 - \pi_0 \right)^{-\frac{1}{\alpha}}, \tag{9}$$

$$\left(\frac{b}{b_0} \right)^* = \lambda_b \left(\pi_0 + (1 - \pi_0) \left(\frac{\lambda_b}{\lambda_a} \frac{\lambda_{a0}}{\lambda_{b0}} \frac{K L_{Y0}}{L_Y K_0} \right)^{-\frac{\alpha\theta}{\alpha - \theta}} \right)^{-\frac{1}{\alpha}}, \tag{10}$$

where $\frac{\alpha\theta}{\alpha - \theta}$ is substituted with $-\alpha$ in the case of Leontief LPFs ($\theta = -\infty$). Inserting these optimal technology choices into the LPF, the GPF takes the normalized CES form:

$$Y = Y_0 \left(\pi_0 \left(\lambda_b \frac{K}{K_0} \right)^\xi + (1 - \pi_0) \left(\frac{\lambda_a}{\lambda_{a0}} \frac{L_Y}{L_{Y0}} \right)^\xi \right)^{\frac{1}{\xi}} \tag{11}$$

where $\xi = \frac{\alpha\theta}{\alpha - \theta}$ and $\xi = \frac{\sigma_{GPF} - 1}{\sigma_{GPF}}$, σ_{GPF} being the aggregate or global elasticity of factor substitution. Henceforth we normalize $\lambda_{b0} = 1$ without loss of generality. As $\xi > \theta$, it is

[22] Second order conditions require us to assume that $\alpha > \theta$, so that the interior stationary point of the above optimization problem is a maximum.

[23] The price of the final good at every point is assumed to be unity.

found that the aggregate elasticity of substitution between capital and labor is uniformly above the elasticity of substitution at the level of each technology.

This contrasts with the Jones (2005b) model, where independent Pareto distributions of unit factor productivities were assumed, leading to aggregate Cobb-Douglas production. The key point is that the current setup preserves a non-unitary elasticity of substitution upon aggregation, which is one of the essential elements of the posited endogenous growth model.

4.3 R&D

We assume that new, factor-augmenting innovations are created endogenously by the respective R&D sectors (Acemoglu, 2003), augmenting the technology menu by increasing the underlying parameters λ_b and λ_a. The two R&D technologies are:

$$\dot{\lambda}_b = A\left(\lambda_b^{1-\omega} x^{\beta} \ell_b^{\nu_b}\right) - d\lambda_b, \tag{12}$$

$$\dot{\lambda}_a = B\left(\lambda_b^{\phi} x^{\eta} \ell_a^{\nu_a}\right)\lambda_a, \tag{13}$$

where ℓ_a and ℓ_b are the shares (or "research intensity") of population employed in labor- and capital-augmenting R&D, respectively, with $\ell_a + \ell_b + \ell_Y = 1$, and $\ell_Y L = L_Y$ (L is total employment). The parameters $\nu_a, \nu_b \in (0, 1]$ represent duplication externalities in R&D (Jones, 2002). Thus, doubling the number of researchers looking for ideas may lead to less than double the number of unique discoveries, Jones and Williams (2000).

The term $x \equiv \frac{\lambda_b k}{\lambda_a}$ captures the technology-corrected degree of capital-augmentation of the workplace, the "lab equipment" term. It is going to be constant along the BGP. The long-term endogenous growth engine is located in the linear labor-augmenting R&D equation. To fulfill the requirement of existence of a balanced growth path along which $\hat{\lambda}_a$ and $\hat{\lambda}_b$ are constant, we assume that $\beta\phi + \eta\omega \neq 0$. Note, there is a decay of capital-augmenting developments at rate $d > 0$, but none for labor-augmenting developments; this can capture the particular obsolescence and embodied character of capital-augmenting technologies, Solow (1960).

Note, the general form equation (12) is akin to Jones' (1995) formulation, generalized by adding obsolescence and the lab equipment term; thus $d = \beta = 0$ retrieves Jones' original specification. And equation (13) is as in Romer (1990) but scale-free (it features a term in ℓ_b instead of $\ell_b \cdot L$) and with lab equipment and a direct spillover from λ_b; setting $\eta = \phi = 0$ retrieves the scale-free version of Romer (1990), cf. Jones (1999).

4.4 The social planner's problem

The social planner maximizes the representative household's utility from discounted consumption, given standard CRRA preferences subject to the budget constraint (15) (i.e., the equation of motion of the aggregate capital stock in per-capita terms), the two R&D technologies (16)–(17), the labor-market clearing condition, (18), and the production function, (19):[24]

$$\max \int_0^\infty \frac{c^{1-\gamma}-1}{1-\gamma} e^{-(\rho+n)t} dt \qquad s.t. \tag{14}$$

$$\dot{k} = y - c - (\delta + n)k - \zeta \dot{a}, \tag{15}$$

$$\dot{\lambda}_b = A\left(\lambda_b^{1-\omega} x^\beta \ell_b^{\nu_b}\right) - d\lambda_b, \tag{16}$$

$$\dot{\lambda}_a = B\left(\lambda_b^\phi x^\eta \ell_a^{\nu_a} \lambda_a\right), \tag{17}$$

$$1 = \ell_a + \ell_b + \ell_Y, \tag{18}$$

$$y = y_0 \left(\pi_0 \left(\lambda_b \frac{k}{k_0}\right)^\xi + (1-\pi_0)\left(\frac{\lambda_a}{\lambda_{a0}} \frac{\ell_Y}{\ell_{Y0}}\right)^\xi\right)^{1/\xi} \tag{19}$$

where $y = Y/L$, $y_0 = Y_0/L_0$. And where $\gamma > 0$ is the inverse of the intertemporal elasticity of substitution, $\rho > 0$ is the rate of time preference, and $n > 0$ is the (exogenous) growth rate of the labor supply. The last term in (15) captures the negative externality that arises from implementing new labor-augmenting technologies, with $\zeta \geq 0$. Since workers need to develop skills compatible with each new technology, it is assumed that there is an external capital cost of such technology shifts (costs of personnel training, learning-by-doing, etc.).

4.5 Decentralized allocation

Let us now proceed to a discussion of the decentralized allocation of the considered model. The general equilibrium shall be obtained as an outcome of the interplay between: households; final goods producers; aggregators of bundles of capital- and labor-intensive intermediate goods; monopolistically competitive producers of differentiated capital- and labor-intensive intermediate goods; and competitive capital- and labor-augmenting R&D firms. We discuss these agents in turn in the following sections.

The construction of the decentralized allocation draws from Romer (1990), Acemoglu (2003), and Jones (2005a). In particular, we use the Dixit-Stiglitz monopolistic competition setup and the increasing variety framework of the R&D sector.

4.5.1 Households

We assume that the representative household maximizes discounted CRRA utility:

$$\max \int_0^\infty \frac{c^{1-\gamma}-1}{1-\gamma} e^{-(\rho-n)t} dt \tag{20}$$

[24]There are three control variables, c, ℓ_a, ℓ_b and three state variables, k, λ_a, λ_b, in this optimization problem.

subject to the budget constraint:

$$\dot{v} = (\tilde{r} - n)v + w - c, \tag{21}$$

where $v = V/L$ is the household's per-capita holding of assets, $V = K + p_a\lambda_a + p_b\lambda_b$. The representative household is the owner of all capital and also holds the shares of monopolistic producers of differentiated capital- and labor-intensive intermediate goods. Capital is rented at a net market rental rate equal to the gross rental rate less depreciation: $r - \delta$. Solving the household's optimization problem yields the familiar Euler equation:

$$\hat{c} = \frac{r - \delta - \rho}{\gamma}, \tag{22}$$

where $\hat{c} \equiv \dot{c}/c$ etc.

4.5.2 Final goods producers

The role of final goods producers is to generate the output of final goods (which are then either consumed by the representative household or saved and invested, leading to physical capital accumulation), taking bundles of capital- and labor-intensive intermediate goods as inputs. They operate in a perfectly competitive environment, where both bundles are remunerated at market rates p_K and p_L, respectively.

The final goods producers operate a normalized CES technology:

$$Y = Y_0 \left(\pi_0 \left(\frac{Y_K}{Y_{K0}} \right)^{\xi} + (1 - \pi_0) \left(\frac{Y_L}{Y_{L0}} \right)^{\xi} \right)^{\frac{1}{\xi}}. \tag{23}$$

The first order condition implies that final goods producers' demand for capital- and labor-intensive intermediate goods bundles satisfies:

$$p_K = \pi \frac{Y}{Y_K}, \; p_L = (1 - \pi) \frac{Y}{Y_L}, \tag{24}$$

where share term $\pi = \pi_0 \left(\frac{Y_K}{Y_{K0}} \frac{Y_0}{Y} \right)^{\xi}$ is the elasticity of final output with respect to Y_K.

4.5.3 Aggregators of capital- and labor-intensive intermediate goods

There are two symmetric sectors in the economy, whose role is to aggregate the differentiated (capital- or labor-intensive) goods into the bundles Y_K and Y_L demanded by final goods producers. It is assumed that the differentiated goods are imperfectly substitutable (albeit gross substitutes). The degree of substitutability is captured by parameter $\varepsilon \in (0, 1)$:

$$Y_K = \left(\int_0^{N_K} X_{Ki}^{\varepsilon} di \right)^{\frac{1}{\varepsilon}}. \tag{25}$$

Aggregators operate in a perfectly competitive environment and decide upon their demand for intermediate goods, price of which will be set by the respective monopolistic producers (discussed in the following subsection).

For capital-intensive bundles, the aggregators maximize:

$$\max_{X_{ki}} \left\{ p_K \left(\int_0^{N_K} X_{Ki}^{\varepsilon} di \right)^{\frac{1}{\varepsilon}} - \int_0^{N_K} p_{Ki} X_{Ki} di \right\}. \tag{26}$$

As we see, there is a continuum of measure N_K of capital-intensive intermediate goods producers. Optimization implies the following demand curve:

$$X_{Ki} = x_K(p_{Ki}) = \left(\frac{p_{Ki}}{p_K} \right)^{\frac{1}{\varepsilon-1}} Y_K^{\frac{1}{\varepsilon}}. \tag{27}$$

Symmetrically, there is also a continuum of measure N_L of labor-intensive intermediate goods producers. The demand curve for their products satisfies

$$X_{Li} = x_L(p_{Li}) = \left(\frac{p_{Li}}{p_L} \right)^{\frac{1}{\varepsilon-1}} Y_L^{\frac{1}{\varepsilon}}. \tag{28}$$

4.5.4 Producers of differentiated intermediate goods

It is assumed that each of the differentiated capital- or labor-intensive intermediate goods producers, indexed by $i \in [0, N_K]$ or $i \in [0, N_L]$ respectively, has monopoly over its specific variety. It is therefore free to choose its preferred price p_{Ki} or p_{Li}. These firms operate a simple linear technology, employing either only capital or only labor.

For the case of capital-intensive intermediate goods producers, the production function is $X_{Ki} = K_i$. Capital is rented at the gross rental rate r. The optimization problem is:

$$\max_{p_{Ki}} (p_{Ki} X_{Ki} - r K_i) = \max_{p_{Ki}} (p_{Ki} - r) x_K(p_{Ki}). \tag{29}$$

The optimal solution implies $p_{Ki} = r/\varepsilon$ for all $i \in [0, N_K]$. This implies symmetry across all differentiated goods: they are sold at equal prices, and thus their supply is also identical, $X_{Ki} = \bar{X}_K$ for all i. Given this regularity, market clearing implies:

$$K = \int_0^{N_K} K_i di = \int_0^{N_K} X_{Ki} di = N_K \bar{X}_K \qquad Y_K = N_K^{\frac{1-\varepsilon}{\varepsilon}} K. \tag{30}$$

Finally, the demand curve implies that the price of intermediate goods is linked to the price of the capital-intensive bundle as in $p_K = p_{Ki} N_K^{\frac{\varepsilon-1}{\varepsilon}} = \frac{r}{\varepsilon} N_K^{\frac{\varepsilon-1}{\varepsilon}}$.

Symmetrically, in the labor-intensive sector, the production function is $X_{Li} = L_{Yi}$. Employees are remunerated at the market wage rate w. The total labor supply is given by $L_Y = \ell_Y L = \int_0^{N_L} L_{Yi} di$. Optimization yields $p_{Li} = w/\varepsilon$. By symmetry, we also obtain:

$$L_Y = \int_0^{N_K} X_{Li} di = N_L \bar{X}_L \qquad Y_L = N_L^{\frac{1-\varepsilon}{\varepsilon}} L_Y. \tag{31}$$

The respective prices satisfy $p_L = p_{Li} N_L^{\frac{\varepsilon-1}{\varepsilon}} = \frac{w}{\varepsilon} N_L^{\frac{\varepsilon-1}{\varepsilon}}$.

Finally, aggregating across all the intermediate goods producers, we obtain that their total profits are equal to $\Pi_K N_K = rK\left(\frac{1-\varepsilon}{\varepsilon}\right)$ and $\Pi_L N_L = wL_Y\left(\frac{1-\varepsilon}{\varepsilon}\right)$ for capital- and labor-intensive goods respectively. Streams of profits per person in the representative household are thus $\pi_K = \Pi_K/L$ and $\pi_L = \Pi_L/L$, respectively. Hence, the total remuneration channeled to the capital-intensive sector is equal to $p_K Y_K = \frac{r}{\varepsilon}K = rK + \Pi_K N_K$, whereas the total remuneration channeled to the labor-intensive sector is equal to $p_L Y_L = \frac{w}{\varepsilon} L_Y = rL_Y + \Pi_L N_L$.

Comparing these results to the optimization problem of the final goods firms leads to,

$$r = \quad \varepsilon\pi\frac{Y}{K} \quad = \varepsilon\pi_0\left(\frac{Y}{K}\right)^{1-\xi}\left(\frac{Y_0}{K_0}\right)^{\xi}\left(\frac{N_K}{N_{K0}}\right)^{\xi\left(\frac{1-\varepsilon}{\varepsilon}\right)}, \tag{32}$$

$$w = \varepsilon(1-\pi)\frac{Y}{L_Y} \quad = \varepsilon(1-\pi_0)\left(\frac{Y}{L_Y}\right)^{1-\xi}\left(\frac{Y_0}{L_{Y0}}\right)^{\xi}\left(\frac{N_L}{N_{L0}}\right)^{\xi\left(\frac{1-\varepsilon}{\varepsilon}\right)}, \tag{33}$$

$$\frac{p_K}{p_L} = \frac{\pi}{1-\pi}\frac{Y_L}{Y_K} \quad = \frac{\pi}{1-\pi}\frac{L_Y}{K}\left(\frac{N_L}{N_K}\right)^{\frac{1-\varepsilon}{\varepsilon}} = \frac{r}{w}\left(\frac{N_L}{N_K}\right)^{\frac{1-\varepsilon}{\varepsilon}}. \tag{34}$$

In equilibrium, factor shares then amount to,

$$\pi = \pi_0\left(\frac{KY_0}{YK_0}\right)^{\xi}\left(\frac{N_K}{N_{K0}}\right)^{\xi\left(\frac{1-\varepsilon}{\varepsilon}\right)}, \tag{35}$$

$$1-\pi = (1-\pi_0)\left(\frac{L_Y Y_0}{Y L_{Y0}}\right)^{\xi}\left(\frac{N_L}{N_{L0}}\right)^{\xi\left(\frac{1-\varepsilon}{\varepsilon}\right)}. \tag{36}$$

Hence, the aggregate production function, obtained after incorporating all these choices into (23), and using the definitions $\lambda_b = N_K^{\frac{1-\varepsilon}{\varepsilon}}$ and $\lambda_a = N_L^{\frac{1-\varepsilon}{\varepsilon}}$, reads:

$$Y = Y_0\left(\pi_0\left(\frac{\lambda_b K}{\lambda_{b0}K_0}\right)^{\xi} + (1-\pi_0)\left(\frac{\lambda_a L_Y}{\lambda_{a0}L_{Y0}}\right)^{\xi}\right)^{\frac{1}{\xi}}. \tag{37}$$

We see that it coincides with the aggregate production function (11) present in the social planner allocation.

4.5.5 Capital- and labor-augmenting R&D firms

The role of capital- and labor-augmenting R&D firms is to produce innovations which increase the variety of available differentiated intermediate goods, either N_K or N_L. Patents never expire, and patent protection is perfect. R&D firms sell these patents to the representative household which sets up a monopoly for each new variety. Patent price, p_b or p_a, which reflects the discounted stream of future monopoly profits, is set at the competitive market. There is free entry to R&D.

R&D firms employ labor only: $L_a = \ell_a L$ and $L_b = \ell_b L$ workers are employed in the labor- and capital-augmenting R&D sectors, respectively. There is also an externality from the total physical capital stock in the economy, working through the "lab equipment" term in the R&D production function. Furthermore, the R&D firms perceive their production technology as linear in labor, while in fact it is concave due to duplication externalities.

Incorporating these assumptions and using the familiar notion $x = \frac{\lambda_b k}{\lambda_a}$, capital-augmenting R&D firms maximize:

$$\max_{\ell_b} \left(p_b \dot{N}_K - w\ell_b \right) = \max_{\ell_b} \left((p_b Q_K - w)\ell_b \right), \tag{38}$$

where $Q_K = A\left(\lambda_b^{\frac{\varepsilon}{1-\varepsilon}-\omega} x^\beta \ell_b^{\nu_b-1}\right)\left(\frac{\varepsilon}{1-\varepsilon}\right)$ is treated by firms as a constant in the steady state (Romer, 1990; Jones, 2005a). Analogously, labor-augmenting R&D firms maximize:

$$\max_{\ell_a} \left(p_a \dot{N}_L - w\ell_a \right) = \max_{\ell_a} \left((p_a Q_L - w)\ell_a \right), \tag{39}$$

where $Q_L = B\left(\lambda_a^{\frac{\varepsilon}{1-\varepsilon}} \lambda_b^\phi x^\eta \ell_a^{\nu_a-1}\right)\left(\frac{\varepsilon}{1-\varepsilon}\right)$ is treated as exogenous.

Free entry into both R&D sectors implies $w = p_b Q_K = p_a Q_L$. Purchase of a patent entitles the holders to a per-capita stream of profits equal to π_K and π_L, respectively. While the production of any labor-augmenting varieties lasts forever, there is a constant rate d at which production of capital-intensive varieties becomes *obsolete*. This effect is external to patent holders and thus is not strategically taken into account when accumulating the patent stock.

4.5.6 Equilibrium

We define the *decentralized equilibrium* as the collection of time paths of all the respective quantities: $c, \ell_a, \ell_b, k, \lambda_b, \lambda_a, Y_K, Y_L, \{X_{Ki}\}, \{X_{Li}\}$ and prices r, w, p_K, p_L, $\{p_{Ki}\}, \{p_{Li}\}, p_a, p_b$ such that: (1) households maximize discounted utility subject to their budget constraint; (2) profit maximization is followed by final-goods producers, aggregators and producers of capital- and labor-intensive intermediate goods, and capital- and labor-augmenting R&D firms; (3) the labor market clears: $L_a + L_b + L_Y = (\ell_a + \ell_b + \ell_Y)L = L$; (4) the asset market clears: $V = vL = K + p_a\lambda_a + p_b\lambda_b$, where assets have equal returns: $r - \delta = \frac{\pi_L}{p_a} + \frac{\dot{p}_a}{p_a} = \frac{\pi_K}{p_b} + \frac{\dot{p}_b}{p_b} - d$; and, finally (5), such that the aggregate capital stock satisfies $\dot{K} = Y - C - \delta K - \zeta \dot{a} L$, where the last term is, as previously discussed, an externality term.

4.6 Solving for the social planner allocation

4.6.1 Balanced growth path

Since Uzawa (1961) we have known that any neoclassical growth model – including the one laid out above – can exhibit balanced growth only if technical change is purely labor-augmenting or if production is Cobb-Douglas. Hence, once we presume a CES production function, the analysis of dynamic consequences of technical change, which is not purely labor-augmenting, must be done outside of the BGP.

Along the BGP, we obtain the following growth rate of the key model variables:

$$g = \hat{\lambda}_a = \hat{k} = \hat{c} = \hat{y} = B(\lambda_b^*)^{\phi}(x^*)^{\eta}(\ell_a^*)^{\nu_a}, \tag{40}$$

where stars denote steady-state values.

Ultimately long-run growth is driven by labor-augmenting R&D. This can be explained by the fact that labor is the only non-accumulable factor, and that it is complementary to capital along the aggregate production function. The following variables are constant along the BGP: $y/k, c/k, \ell_a, \ell_b, \lambda_b$ (thus asymptotically there is no capital-augmenting technical change).

4.6.2 Externality term

The externality term in the social planner's optimization problem can be computed using the firms' optimal technology choice. In general, we derive:

$$a = \lambda_a (N(1-\pi_0))^{1/\alpha} \left(\frac{\pi}{\pi_0}\right)^{1/\alpha} \quad \forall t. \tag{41}$$

Hence the formula for $\dot{a}$ becomes rather involved. To make the externality more tractable, we assume that,

$$\frac{\dot{a}}{k} \propto g \left(\frac{\lambda_b}{x} \left(\frac{\pi}{\pi_0}\right)^{1/\alpha} \right) \tag{42}$$

which is an identity at the BGP but no longer so outside of the BGP.

4.6.3 Euler equations

Having set up the Hamiltonian and computed its derivatives, the following Euler equations are obtained for the social planner allocation:

$$\begin{aligned} \hat{c} &= \frac{1}{\gamma}\left(\frac{y}{k}\left(\pi + \frac{1-\pi}{\ell_Y}\left(\frac{\eta \ell_a}{\nu_a} + \frac{\beta \ell_b}{\nu_b}\right)\right) - \delta - \rho\right), && (43)\\ \varphi_1 \hat{\ell}_a + \varphi_2 \hat{\ell}_b &= Q_1, && (44)\\ \varphi_3 \hat{\ell}_a + \varphi_4 \hat{\ell}_b &= Q_2, && (45) \end{aligned}$$

where

$$\varphi_1 = \nu_a - 1 - (1-\xi)\pi\frac{\ell_a}{\ell_Y}, \tag{46}$$

$$\varphi_2 = -(1-\xi)\pi\frac{\ell_b}{\ell_Y} \tag{47}$$

$$\varphi_3 = -(1-\xi)\pi\frac{\ell_a}{\ell_Y} \tag{48}$$

$$\varphi_4 = \nu_b - 1 - (1-\xi)\pi\frac{\ell_b}{\ell_Y}, \tag{49}$$

$$Q_1 = -\gamma\hat{c} - \rho + n + \hat{\lambda}_a\left(\frac{\ell_Y\nu_a}{\ell_a} + 1 - \eta - \beta\frac{\ell_b\nu_a}{\ell_a\nu_b}\right) - \phi\hat{\lambda}_b + ((1-\xi)\pi - \eta)\hat{x} \tag{50}$$

$$\begin{aligned} Q_2 &= -\gamma\hat{c} - \rho + n + \hat{\lambda}_a + \hat{\lambda}_b\left(\frac{\pi}{1-\pi}\frac{\ell_Y\nu_b}{\ell_b} + (\phi+\eta)\frac{\nu_b\ell_a}{\nu_a\ell_b} + \beta\right) + ((1-\xi)\pi - \beta)\hat{x} \\ &+ d\left(\frac{\pi}{1-\pi}\frac{\ell_Y\nu_b}{\ell_b} + (\phi+\eta)\frac{\nu_b\ell_a}{\nu_a\ell_b} - \omega + \beta\right). \end{aligned} \tag{51}$$

A sufficient condition for all transversality conditions to be satisfied is that $(1-\gamma)g + n < \rho$.

4.6.4 Steady state of the transformed system

The steady state of the transformed dynamical system implied by the social planner solution (i.e., in coordinates: $u = c/k, \ell_a, \ell_b, x, \lambda_a$, with auxiliary variables $z = y/k, \pi, g$) satisfies:

$$g = \hat{\lambda}_a = \hat{k} = \hat{c} = \hat{y} = B(\lambda_b^*)^\phi (x^*)^\eta (\ell_a^*)^{\nu_a} \tag{52}$$

$$\gamma g + \delta + \rho = z\left(\pi + \frac{1-\pi}{\ell_Y}\left(\frac{\eta\ell_a}{\nu_a} + \frac{\beta\ell_b}{\nu_b}\right)\right) \tag{53}$$

$$g = z - \zeta\frac{\dot{a}}{k} - u - (\delta + n) \tag{54}$$

$$d = A\left(\lambda_b^{-\omega} x^\beta \ell_b^{\nu_b}\right) \tag{55}$$

$$(1-\gamma)g + n - \rho = d\left(\frac{\pi}{1-\pi}\frac{\ell_Y\nu_b}{\ell_b} + (\phi+\eta)\frac{\nu_b\ell_a}{\nu_a\ell_b} - \omega + \beta\right) \tag{56}$$

$$(1-\gamma)g + n - \rho = -g\left(\frac{\ell_Y\nu_a}{\ell_a} - \eta - \beta\frac{\ell_b\nu_a}{\ell_a\nu_b}\right) \tag{57}$$

$$\frac{\pi}{\pi_0} = \left(\frac{\lambda_b}{\lambda_{b0}}\right)^\xi \left(\frac{z}{z_0}\right)^{-\xi} \tag{58}$$

$$\frac{z}{z_0} = \frac{\lambda_b}{\lambda_{b0}}\left(\pi_0 + (1-\pi_0)\left(\frac{x_0\ell_Y}{x\ell_{Y0}}\right)^\xi\right)^{1/\xi}. \tag{59}$$

This non-linear system of equations will be solved numerically, yielding the unique steady state of the de-trended system, and thus the unique BGP of the model in original variables. All further analysis of the social planner allocation will be based on the (numerical) linearization of the 5-dimensional dynamical system of equations (43)–(45), (15) and (16), taking the BGP equality (40) as given.

4.7 Solving for the decentralized allocation

When solving for the decentralized allocation, we broadly follow the steps carried out in the case of the social planner allocation. We first solve analytically for the BGP of our endogenous growth model and then linearize the implied dynamical system around the BGP.

4.7.1 Balanced growth path

Along the BGP, we obtain the following growth rate of the key model variables:

$$g = \hat{k} = \hat{c} = \hat{y} = \hat{w} = \hat{p}_b = \hat{p}_{Li} = \hat{\lambda}_a = B(\lambda_b^*)^{\phi}(x^*)^{\eta}(\ell_a^*)^{\nu_a}. \tag{60}$$

The following quantities are constant along the BGP: $y/k, c/k, \ell_a, \ell_b, \lambda_b, Y_K/Y, Y_L/Y$ (and thus again, asymptotically there is no capital-augmenting technical change). The following prices are also constant along the BGP: $r, p_a, p_K, p_L, \{p_{Ki}\}$.

4.7.2 Euler equations

Calculations imply that the decentralized equilibrium is associated with the following Euler equations describing the first-order conditions:

$$\begin{aligned} \hat{c} &= \frac{1}{\gamma}\left(\varepsilon\pi\frac{y}{k} - \delta - \rho\right), & (61)\\ \varphi_1\hat{\ell}_a + \varphi_2\hat{\ell}_b &= \tilde{Q}_1, & (62)\\ \varphi_3\hat{\ell}_a + \varphi_4\hat{\ell}_b &= \tilde{Q}_2, & (63) \end{aligned}$$

where

$$\begin{aligned} \tilde{Q}_1 &= -\varepsilon\pi\frac{y}{k} + \delta + \hat{\lambda}_a\frac{\ell_Y}{\ell_a} - \phi\hat{\lambda}_b + ((1-\xi)\pi - \eta)\hat{x}, & (64)\\ \tilde{Q}_2 &= -\varepsilon\pi\frac{y}{k} + \delta + \hat{\lambda}_a + (\hat{\lambda}_b + d)\left(\frac{\pi}{1-\pi}\frac{\ell_Y}{\ell_b}\right) - \hat{\lambda}_b(1-\omega) - d + ((1-\xi)\pi - \beta)\hat{x}, & (65) \end{aligned}$$

and where φ_1 through φ_4 are defined as in equations (46)–(49). A sufficient condition for all transversality conditions to be satisfied is that $(1-\gamma)g + n < \rho$.

4.7.3 Departures from the social optimum

Departures of the decentralized allocation from the social planner's can be tracked back to some specific assumptions regarding the information structure of the decentralized allocation. Those differences are the following:

1. In the consumption Euler equation, the term $\frac{y}{k}\left(\pi + \frac{1-\pi}{\ell_Y}\left(\frac{\eta\ell_a}{\nu_a} + \frac{\beta\ell_b}{\nu_b}\right)\right)$ is replaced by $\varepsilon\pi\frac{y}{k}$. This is due to two effects: (a) in contrast to the social planner, markets fail to account for the external effects of physical capital on R&D activity via the lab

equipment terms (with elasticity β in the case of capital-augmenting R&D, and η in the case of labor-augmenting R&D); (b) ε appears in the decentralized allocation due to imperfect competition in the labor- and capital-augmenting intermediate goods sectors.

2. In the Euler equations for ℓ_a and ℓ_b, the shadow price of physical capital $\hat{c} - \rho + n$ is replaced by its market price $r - \delta = \varepsilon \pi \frac{y}{k} - \delta$ which accounts for markups arising from imperfect competition.

3. In the Euler equation for ℓ_a, the term $\left(\frac{\ell_Y \nu_a}{\ell_a} + 1 - \eta - \beta \frac{\ell_b \nu_a}{\ell_a \nu_b}\right)$ is replaced by $\frac{\ell_Y}{\ell_a}$. This is due to two effects: (a) ν_a is missing because markets fail to internalize the labor-augmenting R&D duplication effects inherent when $\nu_a < 1$; (b) the latter two components are missing because markets fail to account for the external effects of accumulating knowledge on future R&D productivity. These effects are included in the shadow prices of λ_a and λ_b in the social planner allocation but not in their respective market prices.

4. Analogously, in the Euler equation for ℓ_b, the term $\left(\frac{\ell_Y \nu_b}{\ell_b} \frac{\pi}{1-\pi} + (1 - \omega + \beta) + (\phi + \eta) \frac{\ell_a \nu_b}{\ell_b \nu_a}\right)$ is replaced by $\left(\frac{\ell_Y}{\ell_b} \frac{\pi}{1-\pi}\right)$. The same reasoning follows as per point 3.

4.7.4 Steady state of the transformed system: decentralized solution

The steady state of the transformed system satisfies:

$$
\begin{aligned}
g &= \hat{\lambda}_a = \hat{k} = \hat{c} = \hat{y} = B(\lambda_b^*)^{\phi} (x^*)^{\eta} (\ell_a^*)^{\nu_a} && (66) \\
\gamma g + \rho &= r - \delta && (67) \\
g &= z - \zeta \frac{\dot{a}}{k} - u - (\delta + n) && (68) \\
d &= A \left(\lambda_b^{-\omega} x^{\beta} \ell_b^{\nu_b}\right) && (69) \\
g \frac{\ell_Y}{\ell_a} &= r - \delta && (70) \\
g &= r - \delta + d \left(1 - \frac{\pi}{1-\pi} \frac{\ell_Y}{\ell_b}\right) && (71) \\
r &= \varepsilon \pi z && (72) \\
\frac{\pi}{\pi_0} &= \left(\frac{\lambda_b}{\lambda_{b0}}\right)^{\xi} \left(\frac{z}{z_0}\right)^{-\xi} && (73) \\
\frac{z}{z_0} &= \frac{\lambda_b}{\lambda_{b0}} \left(\pi_0 + (1 - \pi_0) \left(\frac{x_0 \ell_Y}{x \ell_{Y0}}\right)^{\xi}\right)^{1/\xi}. && (74)
\end{aligned}
$$

This non-linear system of equations will be solved numerically, yielding the unique steady state of the de-trended system, and thus the unique BGP of the model in original variables. All further analysis of the decentralized allocation will be based on the (numerical) linearization of the 5-dimensional dynamical system of equations (61)–(63), (15) and (16), taking the BGP equality (60) as given.

5 Model solution and dynamics

5.1 Calibration

The model calibration is documented in **Table 7**. It is difficult to calibrate a model such as this given the scarcity of some key data and, more generally, the imperfect mapping between model concepts and observables. To the extent possible, we have relied on benchmark values found in the literature. In any case, we provide an in-depth robustness check of the model by varying those parameters over a wide and plausible support.

The calibration follows five main steps. First, several "deep" parameters are pre-determined by taking values stemming from the associated literature: the intertemporal elasticity of substitution, and time preference. Second, we assign CES normalization parameters to match US long-run averages, spanning 1929–2012 (for factor shares), or post-war estimates (for the aggregate substitution elasticity).[25]

Third, we assume that a range of long-run averages from US data are exactly matched by the BGP of (the decentralized allocation of) the model. Doing so allows us to calibrate economic growth, population growth, capital productivity, and the consumption-to-capital ratio.

Next, with this in hand, four identities included in the system (66)–(74) drive the calibration of other parameters in a model-consistent manner: $\ell_y^*, r^*, \lambda_b^*, x^*$ and ε. Final production employment is set in a model-consistent manner. In the absence of any other information, we agnostically assume that the residual share $1-\ell_Y^*$ is split equally between employment in both R&D sectors.

For the model-consistent value of ℓ_Y^*, this formula leads to relatively high research employment shares. But that may be defended on a number of grounds. The model's limited occupational granularity (i.e., employment is either in R&D or final-goods production), mechanically produces such an outcome.[26] The shares are therefore implicitly proxying for a number of other skills or tasks within and beyond conventionally-classified research activities.

Indeed, counting the number of scientists/researchers/patents has long been recognised as a crude proxy for research activity. In this respect, we might instead interpret the ℓ_a^* and ℓ_b^* values as a correction for the managerial and entrepreneurial input to production as well as learning-by-doing on the side of employees; when new technologies are implemented in production, they require significant effort and/or reorganization of the workplace, which shows up as R&D in our simplified model. Similarly, it may capture non-routine and analytical tasks in the employment spectrum which do not necessarily show up in formal research-intensive job definitions (see Autor, Levy, and Murnane, 2003).

[25] Note, following (11) we can plausibly set $\alpha = 1$ (Growiec, 2013) and thus determine the aggregate elasticity of substitution via θ.

[26] It may be moved up or down to some degree depending on what one assumes for preferences: e.g., as agents become more impatient, less labor is allocated to R&D: $\partial \ell_Y^*/\partial \rho > 0$. The same holds for the inter-temporal substitution elasticity.

Table 7: Baseline calibration: pre-determined parameters

Parameter		Value	Source/Target
Preferences			
Inverse Intertemporal Elasticity of Substitution	γ	1.7500	Barro and Sala-i-Martin (2003)
Time Preference	ρ	0.0200	Barro and Sala-i-Martin (2003)
Income and Production			
GDP Per-Capita Growth	g	0.0171	geometric average of gross growth rates
Population Growth rate	n	0.0153	geometric average of gross growth rates
Labor in Aggregate Production	ℓ_{Y0}, ℓ_Y^*	0.5394	$\frac{0.5\left(\gamma+\frac{\rho}{g}\right)}{1+0.5\left(\gamma+\frac{\rho}{g}\right)}$
Capital Productivity	z_0, z^*	0.3442	geometric average
Consumption-to-Capital	u^*	0.2199	geometric average
Capital Income Share	π_0, π^*	0.3260	arithmetic average
Depreciation	δ	0.0600	Caselli (2005)
Factor Substitution Parameter†	θ	-0.7500	$\sigma = 0.7$, Klump, McAdam, and Willman (2012)
Net Real Rate of Return	$r^* - \delta$	0.0499	$r^* - \delta = \gamma g + \rho$
Substitutability Between Intermediate Goods	ε	0.9793	$\varepsilon = \frac{r^*}{\pi^* z^*}$
R&D Sectors			
R&D Duplication Parameters	$\nu_a = \nu_b$	0.75	see text
Technology-Augmenting Terms	$\lambda_{a0}, \lambda_{b0}$	1.0000	see text
Technology-Augmenting Terms	λ_b^*	1.0000	$\lambda_b^* = \lambda_{b0} \frac{z^*}{z_0} \left(\frac{\pi^*}{\pi_0}\right)^{\frac{1}{\xi}}$
Labor Input in R&D sectors	ℓ_Y^a, ℓ_Y^b	0.2033	$\ell_a^* = \ell_b^*$ for $\ell_a^* + \ell_b^* = 1 - \ell_Y^*$
Lab-Equipment Term‡	x_0, x^*	61.7900	$x^* = x_0 \frac{\ell_Y^*}{\ell_{Y0}} \left(\frac{1}{1-\pi_0}\left(\frac{z^*}{z_0}\frac{\lambda_{b0}}{\lambda_b^*}\right)^{\xi} - \frac{\pi_0}{1-\pi_0}\right)^{-1/\xi}$

Notes: † Thus for $\alpha = 1$, $\xi = \frac{\alpha\theta}{\alpha-\theta} = -0.4$. This is consistent with $\xi = \frac{\sigma-1}{\sigma}$, where the aggregate elasticity of factor substitution $\sigma = 0.7$. ‡ $x_0 = \frac{\lambda_{b0} k_0}{\lambda_{a0}} = 61.79$.

Table 8: Baseline calibration: free parameters.

Parameter		Value
Obsolescence rate of KATC	d	0.0795
Unit productivity of KATC	A	0.1565
Unit productivity of LATC	B	0.0210
Lab equipment in KATC	β	0.1256
Lab equipment in LATC	η	0.2404
Technology choice externality	ζ	115.2793
Degree of DRS in KATC	ω	0.5000
Spillover from KATC to LATC	ϕ	0.3000

As regards the duplication externalities in factor-augmenting R&D, the literature is generally agnostic about their magnitude (Jones, 2002). Moreover, the literature typically considers a *unique* R&D duplication externality.[27] Hence, in our baseline calibration we agnostically set $\nu_a = \nu_b = 0.75$, being an average of the original constant-returns-to-scale parametrization of $\nu = 1$ (Romer, 1990) and some preliminary, shaky empirical evidence of $\nu \approx 0.5$ provided by Pessoa (2005).[28] The technology-augmenting term λ_b^* is set in a model-consistent manner, and following Klump, McAdam, and Willman (2007), we set its normalized value to unity.

The final step is to assign values to the remaining parameters, in particular the technological parameters of R&D equations. We do this by solving the four remaining equations in the system (66)–(74) with respect to eight remaining parameters, see **Table 8**.[29] Since this system is over-identified, there is an infinity of calibrations in agreement with our set of restrictions. Thus, the natural next step is to explore this space of "consistent" calibrations for their implications for model dynamics.

Given the above benchmark calibration, the steady state is a saddle point. Since there are two state-like variables and three (i.e., more than two) stable roots, we conclude that there is local indeterminacy: there are infinitely many (a one-parameter family of) paths of convergence to the steady state which are in agreement with transversality conditions. In the benchmark case, two of the stable eigenvalues are complex, indicating that convergence along the stable manifold is oscillatory.

5.2 Is the decentralized labor share socially optimal?

A familiar outcome in this class of models is that the socially-optimal growth rate exceeds the decentralized one: this is due to the absence of monopolistic markups plus the social

[27] We are not aware of any study which distinguishes between duplication externalities in labor- and capital-augmenting R&D.

[28] In the Appendix, we extensively study the dependence of our results on the assumptions on ν_a and ν_b (see Table C.3).

[29] Note, in his estimations, Pessoa (2005) uses values for the obsolescence parameter between zero and fifteen percent; our endogenously-determined value is thus in the middle of that range.

value of innovations captured by the external effects of capital in R&D. And this is what we see (**Table 9**): the BGP of the decentralized solution features a relatively lower growth rate and excessive consumption (u is higher). With lower growth, capital costs are cheaper and the net real rate of return of capital is higher, and capital productivity is accordingly higher.

Table 9: BGP comparison under the baseline calibration.

Variable		DC	SP
Output growth rate	g	0.0171	0.0339
Consumption-to-capital ratio	u^*	0.2199	0.1628
Capital productivity	z^*	0.3442	0.3071
Employment in production	ℓ_Y^*	0.5934	0.4385
Share of labor-augmenting R&D	ℓ_a^*	0.2033	0.2575
Share of capital-augmenting R&D	ℓ_b^*	0.2033	0.3040
Capital income share	π^*	0.3261	0.2146
Labor income share	$1-\pi^*$	0.6739	0.7854
Net real rate of return	$r^*-\delta$	0.0499	0.0059
Capital-augmenting technology	λ_b^*	1.0000	2.3696
Lab equipment	x^*	61.7900	173.3363

One striking result is that the BGP labor share in the decentralized equilibrium is 11 percentage points below the social optimum. A series of robustness checks (**Figure 4**) confirms that such a large discrepancy is a remarkably robust result.[30]

Moreover, it is straightforward to demonstrate that, at any point in time, the capital income share can be expressed as either of the two following equations:

$$\frac{\pi}{\pi_0} = \left(\frac{\lambda_b k}{k_0}\right)^{\xi}\left(\frac{y}{y_0}\right)^{-\xi} \quad \Rightarrow \quad \hat{\pi} = \xi(\hat{\lambda}_b + \hat{k} - \hat{y}), \tag{75}$$

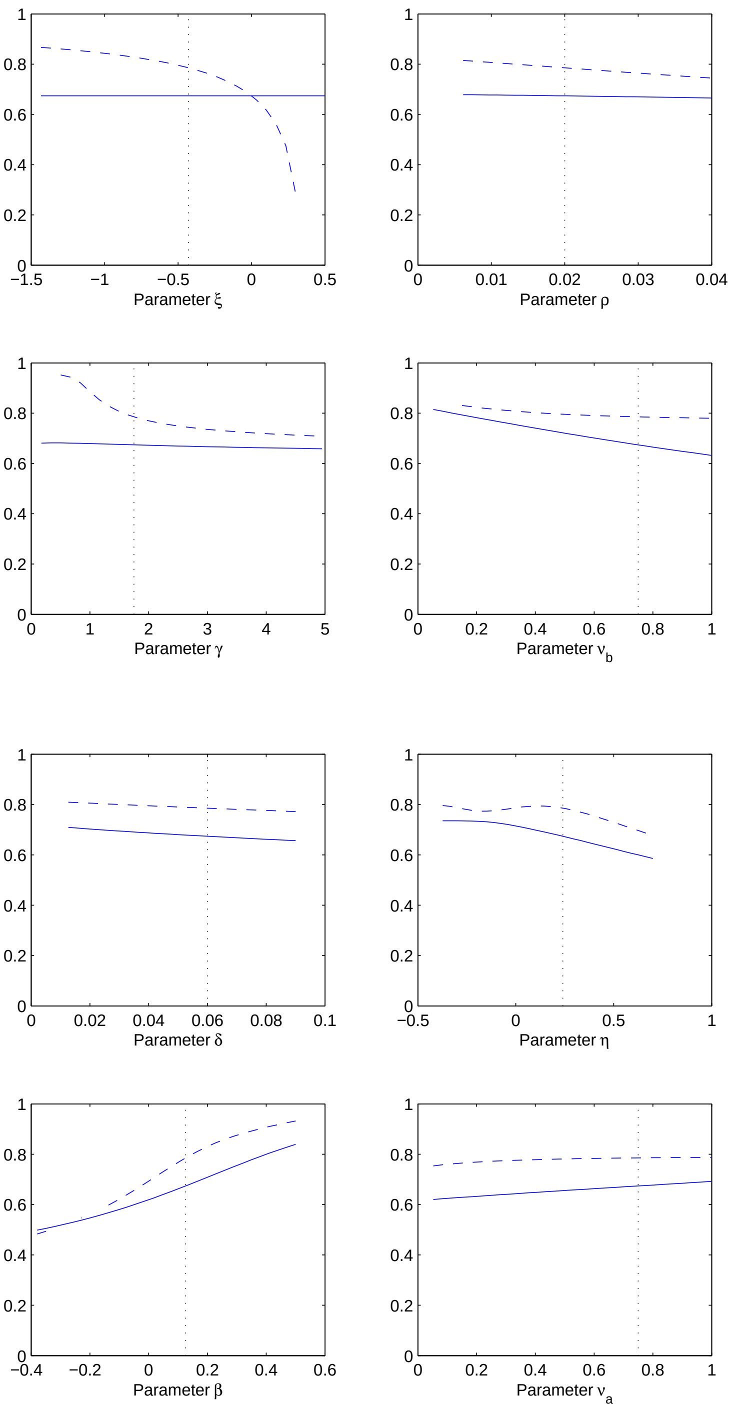
$$\frac{\pi}{1-\pi} = \frac{\pi_0}{1-\pi_0}\left(\frac{x}{x_0}\frac{\ell_{Y0}}{\ell_Y}\right)^{\xi} \quad \Rightarrow \quad \hat{\pi} = \xi(1-\pi)(\hat{x} - \hat{\ell}_Y). \tag{76}$$

Equation (75) shows that differences in the capital income share are driven by levels of capital augmentation and capital productivity. Under gross complementarity ($\xi < 0$), the capital share (labor share) decreases (increases) with both capital augmentation and capital deepening.

Equation (76), in turn, follows from the definitions of the aggregate production function and the "lab equipment" term x. Given $\hat{\ell}_Y \equiv -\left(\frac{\ell_a}{\ell_Y}\hat{\ell}_a + \frac{\ell_b}{\ell_Y}\hat{\ell}_b\right)$, the dynamics of employment

[30] **Figure 4** reveals that the only case where the decentralized allocation leads to a relatively higher labor share is when capital and labor are gross substitutes ($\xi > 0$). Note that the lack of dependence of the BGP on ξ in the decentralized allocation follows from CES normalization (Klump and de La Grandville, 2000), coupled with the fact that we have calibrated the normalization constants to the BGP of the decentralized allocation. A more extensive study of the dependence of both BGPs on key model parameters (ξ, ρ, γ, ν_b) is included in the Appendix.

Figure 4: Dependence of the equilibrium labor share on the model parametrization.



Notes: $1-\pi$ on vertical axis; relevant parameter support on the horizontal axis. Social planner allocation (dashed lines), decentralized equilibrium (solid lines). Vertical dotted line represents the calibrated value.

in the goods sector are equal to the inverse of the dynamics of total R&D employment. It then follows that dynamics of the labor share are uniquely determined by the sum of the dynamics of the lab equipment component and R&D employment. Likewise, the sign of this relationship depends on the substitution elasticity: if $\xi < 0$ (gross complementarity) then increases in R&D intensity reduce π, and thus increase the labor share, and vice versa.

Comparing the decentralized and the social planner's allocation through the lens of equation (75), we observe that the large difference in factor shares at the BGP is driven almost exclusively by the difference in the level of capital augmentation λ_b^*. This result underscores our initial hypothesis that technical change is quantitatively more important for explaining the labor share than capital deepening.

Equivalently, by equation (76), this large difference in the degree of capital augmentation shows up in the lab equipment term x^*. It is also further strenghened by the discrepancy in employment in production ℓ_Y^*, which is higher in the decentralized allocation because the planner devotes more resources to (both types of) R&D. Thanks to this, coupled with relatively more saving, it achieves faster growth at the BGP but with a lower consumption-to-capital ratio and a lower capital's rate of return.

Table 10: Dynamics around the BGP under the baseline calibration.

Allocation	DC	SP
Pace of convergence (% per year)	6.3%	4.2%
Length of full cycle (years)	52.6	76.7

Given our baseline calibration, both allocations exhibit endogenous, dampened oscillations of the labor share and other de-trended model variables **(Table 10)**. The decentralized allocation features relatively shorter cycles but also faster convergence to the BGP. Hence, it cannot be claimed directly that the decentralized equilibrium has *excessive volatility*. If both allocations were to start from the same initial point outside of the BGP then, absent random shocks, the decentralized allocation would exhibit a greater frequency but smaller amplitude of cyclical variation.

Having scrutinized the robustness of this result by altering the model parametrization (Table C.3 in the Appendix), we conclude that while the decentralized equilibrium generally exhibits shorter cycles, the ordering of both allocations in terms of pace of convergence can sometimes be reversed. This finding lends partial support to the claim that the decentralized equilibrium is likely to feature (a bit) too much labor share volatility as compared to the social optimum.

On the last point, note that even though our study does not intend to match the frequency of medium-to-long swings exactly, we still view the R&D-based endogenous growth model with CES production as a viable explanation for the hump-shaped trend of the labor share observed in the US throughout the twentieth century. In fact, if one argued that in our 83-year time series of the US labor share, we observe only a half of a full long-run swing, then the model could match that exactly if the imaginary parts of stable roots were around

0.04 which could be obtained e.g., for lower duplication parameters ν_a and ν_b, a higher lab equipment term β or lower η, etc.

5.3 Emergence of stable limit cycles via Hopf bifurcations

Let us now turn to the question if one could plausibly expect limit cycle behavior of the labor share. At the baseline calibration, the answer is negative because the model exhibits oscillatory convergence to the BGP – which precludes limit cycles. On the other hand, we know that by Hopf's bifurcation theorem (see e.g. Feichtinger, 1992), if when manipulating one of the model parameters, real parts of two stable conjugate roots of the linearized system around the steady state transversally cross zero, the steady state loses its stability and a stable limit cycle is created around it.[31] The question is then, does such a situation appear in our case, and if so, does it happen for an empirical plausible set of model parameters?

Moreover, inspection of eigenvalues of the linearized system reveals that apart from Hopf bifurcations, one further interesting type of sudden changes in model dynamics can be observed: a "node-focus" bifurcation. If, manipulating one of the model parameters, two stable real eigenvalues collide and become complex conjugates, then pattern of convergence to the steady state changes from monotonic to oscillatory.

The current numerical exercise has been carried out as follows. We computed the eigenvalues of the system of dynamic model equations around the steady state of the model for various values of one certain parameter in question, assuring that whatever assumption on its value is made, other "free" parameters are re-calibrated in a way that the model always remains in perfect accordance with BGP characteristics based on US data. In this way, we are able to identify model parametrizations leading to various types of local dynamics around *the same* BGP.

Figures 5–6 illustrate the results of such "multi-calibration" exercise (or "BGP-preserving" sensitivity analysis) for the following parameters: ρ (the rate of time preference), γ (the inverse elasticity of intertemporal substitution in consumption), $\frac{\ell_a^*}{\ell_a^*+\ell_b^*}$ (the share of labor-augmenting R&D at the BGP) as well as ξ (and thus the elasticity of substitution) and β (the lab equipment term in capital-augmenting R&D).[32] *Hopf bifurcations*, where a stable limit cycle around the steady state is created, are obtained when manipulating the three former parameters: ρ (the rate of time preference), γ (the inverse elasticity of intertemporal substitution in consumption) and $\frac{\ell_a^*}{\ell_a^*+\ell_b^*}$ (the share of labor-augmenting R&D at the BGP). Specifically, real parts of a pair of conjugate eigenvalues cross zero when the time preference parameter ρ becomes sufficiently small (below 0.0062 per annum in the baseline case), or when the intertemporal elasticity of substitution becomes sufficiently large (γ below 0.9430).

[31]More precisely, in the multi-dimensional case with which we are dealing here, at the point of a Hopf bifurcation the steady state ceases to be a stable focus along the stable manifold, and becomes a repelling focus instead, and a stable limit cycle around the steady state is created in the subspace formerly referred to as the stable manifold.

[32]Results for the other parameters – δ, η, ν_b, and ν_a – have been relegated to the appendix as they are not conducive to qualitative changes in model dynamics.

We conclude that if the representative household is sufficiently patient, or sufficiently willing to substitute consumption across time, not only does the pace of convergence to the steady state slow down, but also – owing to the Hopf bifurcation theorem – stable limit cycles around the steady state can be created. We also note that when the steady-state ratio of labor-augmenting R&D to total R&D employment is sufficiently small (around 0.41), a stable limit cycle is created as well.

These numbers are clearly in the ballpark of empirically plausible values. To get a limit cycle in factor shares (as well as other model parameters), it suffices that the household is just slightly more willing to substitute consumption across time than in the case of logarithmic preferences. Although the critical value of the time preference rate is rather low in the baseline case, it should be noted that with $\gamma = 1$ (log preferences), the bifurcation value with respect to ρ appears already around 0.0190, and thus it is in the immediate vicinity of its baseline value of 0.02. Even more curiously, we demonstrate (**Figure C.7** in the Appendix) that with a somewhat modified parametrization, with asymmetric R&D duplication externalities, $\nu_a = 0.5$ and $\nu_b = 0.9$,[33] Hopf bifurcations are even more clearly visible, and stable limit cycles appear even closer to the baseline. In such case, the critical value of γ increases to 1.0794, and thus log preferences are then already in the area of limit cycles. The critical value of ρ rises to 0.0086.

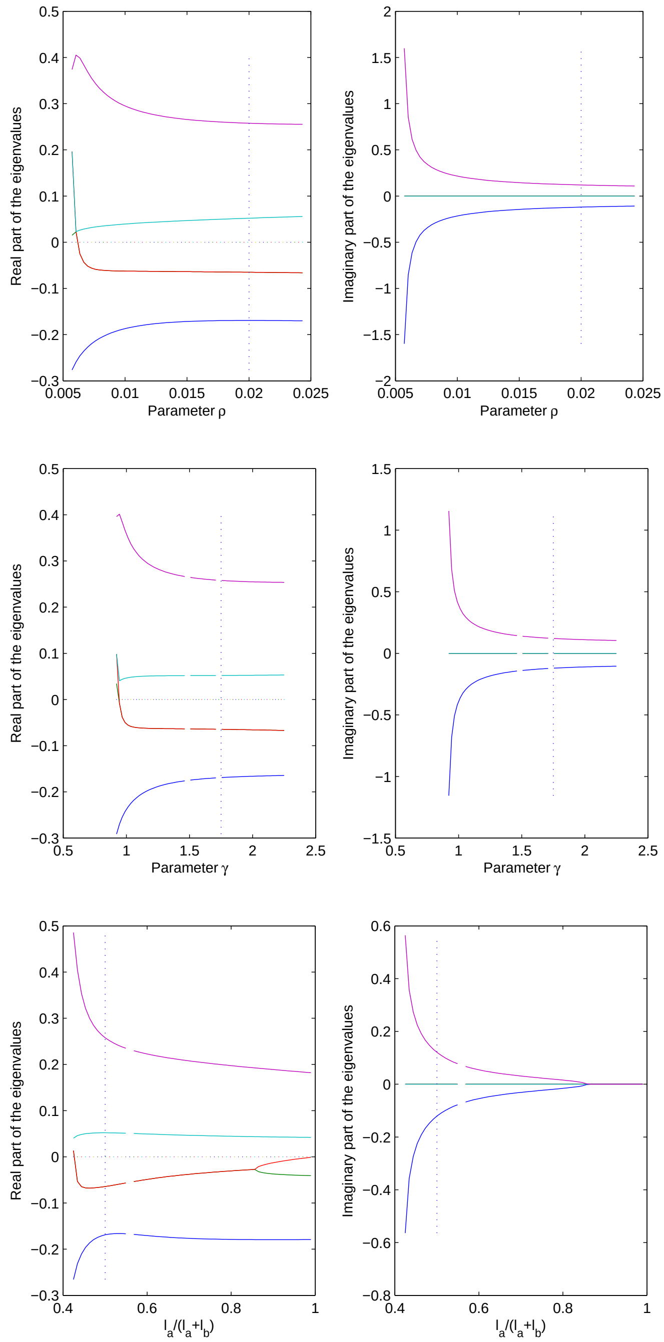
"Node–focus" bifurcations, on the other hand, are found when manipulating ξ (the factor substitutability parameter), β (the "lab equipment" term in capital-augmenting technical change), and again $\frac{\ell_a^*}{\ell_a^*+\ell_b^*}$. Specifically, we observe that when ξ becomes sufficiently large (and already above zero so that we are in the range of gross substitutability of inputs), then the imaginary parts of two conjugate stable roots hit zero, so that oscillatory dynamics are eliminated. Under gross complementarity, the magnitude of input complementarity generally increases the frequency of observed oscillations. Our another finding is that β, measuring the impact of lab equipment on the productivity of capital-augmenting R&D, is also conducive to "node–focus" bifurcations. Oscillatory dynamics prevail only if β is sufficiently low, and the lower it is, the higher the oscillation frequency. By the same token, when the steady-state share of labor-augmenting R&D in total R&D employment is sufficiently high (above 90%), dampened oscillations disappear in favor of monotonic convergence.

Finally, as far as manipulations in the parameter ξ are concerned, we also observe a further phenomenon. Namely, in the range of gross substitutability there exists a critical value of ξ when a real root switches its sign. At this point a *generalized saddle-node bifurcation* appears, due to which the dynamics around the steady state (of the de-trended system) switch from locally indeterminate to fully determinate. If ξ is above a specific threshold value then there exists a unique saddle path, along which convergence to the steady state is monotonic.

As there are no zeros in the implied eigenvectors (available upon request), we infer that (locally) all original variables (including the labor share $1 - \pi$) of the model oscillate when

[33] This captures the possibility that R&D duplication externalities are stronger in labor-augmenting R&D, as there is greater scope for patent protection when the technology is embodied in capital goods and subject to obsolescence (see the discussion in Solow, 1960).

Figure 5: Emergence of limit cycles via Hopf bifurcations.

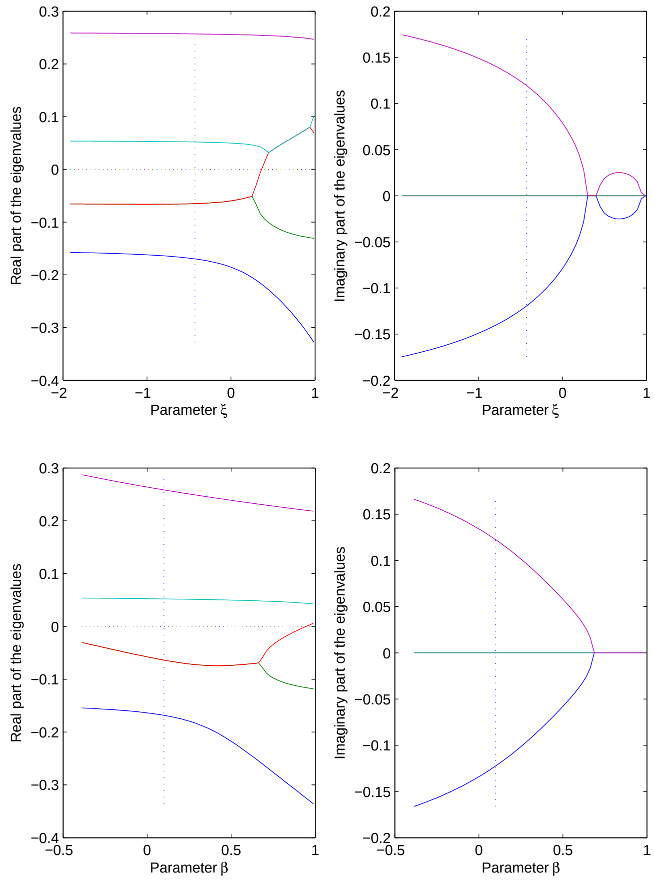


converging to the steady state, with the same frequency of oscillations. Hence, we conclude that the following variables exhibit oscillations of the same frequency as the labor share:

(a) the consumption–capital ratio, c/k,

(b) the output–capital ratio, y/k,

Figure 6: "Node–focus" bifurcations.



(c) the growth rate of the economy, g,

(d) the share of employment in capital-augmenting R&D, ℓ_b,

(e) the share of employment in labor-augmenting R&D, ℓ_a,

(f) the unit factor productivity of capital along the aggregate production function, λ_b,

(g) the "lab equipment" term, i.e., technology-corrected degree of capital-augmentation of the workplace, x.

Finally, as demonstrated in the Appendix **(Figures C.5–C.6)**, the pattern of dependence of model dynamics on its parameters is generally quite similar in the case of the decentralized equilibrium and the social planner allocation, and differences are only quantitative. This suggests that emergence of stable limit cycles via Hopf bifurcations as well as "node-focus" bifurcations can occur in both cases.[34]

[34]Note that the bifurcation graphs in **Figures C.5–C.6** are non-BGP-preserving. Unlike **Figures 5–6**, the model has not been sequentially re-calibrated here for to keep the BGP intact. The reason for this change

6 Conclusions

The contribution of the current article to the literature has been (i) to document that the labor's share of GDP in fact exhibits substantial medium-run swings, and (ii) to provide a theoretical assessment of the extent a calibrated endogenous growth model can account for these regularities.

To our knowledge, this is the first paper to emphasize medium-term swings of the labor share, focusing on their technological explanation, and allowing the possibility that these swings be driven by endogenous, stable limit cycles. We keep our model setup relatively simple, focusing exclusively on endogenous growth mechanisms, and treating our model as a laboratory for testing to which extent these mechanisms are able to explain the observed patterns of labor share swings.

Our workhorse model has been set up as a micro-founded endogenous growth model with aggregate CES technology and directed capital- and labor-augmenting technical change à la Acemoglu (2003). We computed both the decentralized and socially optimal solution and analysed its dynamics. Having calibrated the parameters of our model based on US data, we have carried out a sequence of numerical exercises allowing us to:

- confirm that the interplay between endogenous arrivals of capital- and labor-augmenting technologies leads to oscillatory convergence to the long-run growth path as well as stable limit cycles via Hopf bifurcations, and
- assess the magnitude of departures of the decentralized allocation from the first best.

More specifically, our theoretical model has the following implications.

- The model provides an explanation for long swings in the labor share: it implies oscillatory dynamics of factor shares along the convergence path to the BGP.
- Under certain empirically plausible parametrizations, the model gives rise to Hopf bifurcations, indicating the emergence of limit cycles in the labor share.
- Oscillatory behavior of the labor share is due to the interplay between arrivals of capital-augmenting developments (subject to gradual depreciation) and changes in the pace of (continued) labor-augmenting technical change.

in approach is that a different re-calibration would be necessary for each of the two allocations, and that would ruin their comparability.

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A Data construction

A.1 Labor share

The broadly used approach in measuring the labor share is simply dividing Compensation of Employees (CE) by the GDP. But that does not take into consideration the self-employees income. Unfortunately, self-employees labor income is published with the capital income. Since Gollin (2002) a few adjustments have been proposed. We incorporate one of the most detailed way in the measuring labor share suggested by Gomme and Rupert (2007); Ríos-Rull and Santaeulàlia-Llopis (2010) and which takes into consideration the unknown (self-employees) income. The starting point is the assumption that proportion of the unknown labor (capital) income to the total unknown income is the same as the ratio of known labor (capital) to the known income of both share. The unknown income (AI) is the sum of Proprietor's Income (PI), Business Current Transfer Payments ($BCTP$), Statistical Discrepancy ($SDis$) and Taxes on Production (Tax) reduced by Subsidies (Sub) ($AI = PI + Tax - Sub + BCTP + SDis$). On the other hand, known capital income (UCI) consists of Rental Income (RI), Current Surplus of Government Enterprises (GE), Net Interests (NI) and Corporate Profits (CP). Including UCI to Compensation of Employees (CE) and consumption of fixed capital (DEP) we derive total unambiguous income (UI) and can calculate the portion of UCI to UI: $\theta = \frac{UCI + DEP}{UI}$. Having θ it is easy to obtain ambiguous capital income (ACI) which equals $AC \cdot \theta$. Finally, we derive labor share income as one minus capital income share:

$$LS_t = 1 - \frac{UCI + DEP + ACI}{GDP} = 1 - \theta \tag{A.1}$$

GDP and Consumption of fixed capital (DEP) are taken from NIPA*[Table 1.7.5]* and the rest series are taken from NIPA *[Table 1.12]*.

A.2 Macroeconomic variables

GDP - Gross Domestic Product in billions of chained (2005) dollars, BEA NIPA Table 1.6.
Labor productivity (LP_t) - Labor Productivity in nonfarm business sector, index (2009=100), BLS Series No. PRS85006093.
Consumption (C_t) - Personal consumption expenditures in billions of chained (2005) dollars, BEA NIPA Table 1.6.
Investment (I_t) - Gross private domestic investment in billions of chained (2005) dollars, BEA NIPA Table 1.6.
Government expenditures (G_t) - Government consumption expenditures and gross investment in billions of chained (2005) dollars, BEA NIPA Table 1.6.
Consumption to product ratio (C_t/Y_t) - Personal consumption expenditures (in billions of current dollars) divided by Gross Domestic Product (in billions of current dollars), both series from BEA NIPA Table 1.5.
Investment to product ratio (I_t/Y_t) - Gross private domestic investment (in billions of current dollars) divided by Gross Domestic Product (in billions of current dollars), both series from BEA NIPA Table 1.5.
Consumption of durables goods to product ratio (C_t^{DG}/Y_t) - Personal consumption expenditures on durables goods (in billions of current dollars) divided by Gross Domestic Product (in billions of current dollars), both series from BEA NIPA Table 1.5.
Consumption of non-durables goods to product ratio (C_t^{NDG}/Y_t) - Personal consumption expenditures on non-durables goods (in billions of current dollars) divided by Gross Domestic Product (in billions of current dollars), both series from BEA NIPA Table 1.5.

R&D expenditures (RD_t) – Research and development expenditures in constant millions of dollars, series taken from National Science Foundation.

The share of the R&D expenditures in Gross Domestic Product (RD_t/GDP_t) is ratio of total spending on research and development sector divided by GDP, both series in current millions of dollars, taken from National Science Foundation and BEA NIPA Table 1.5, respectively.

Employment (E_t) - Employment in non-farm business sector, index (2009=100), BLS Series No. PRS85006013.

Hours (H_t) - Hours in non-farm business sector, index, BLS Series No. PRS85006033.

Consumption to private physical capital stock (C_t/K_t) - ratio of consumption expenditures in current billions of dollars to private fixed assets stock also in current billions of dollars, series taken from BEA NIPA Table 1.5 and BEA Fixed Assets Table 1.1 respectively.

Private capital stock to product ($K_t^{PRIVATE}/GDP_t$) - private fixed assets stock in current billions of dollars to Gross Domestic Product, series taken from BEA Fixed Assets Table 1.1 and BEA NIPA Table 1.5 respectively, time span.

Aggregate hours of unskilled workers ($hours_t^U$) - the aggregated hours in all sectors for all employees with education not higher than college, raw data taken from World KLEMS database.

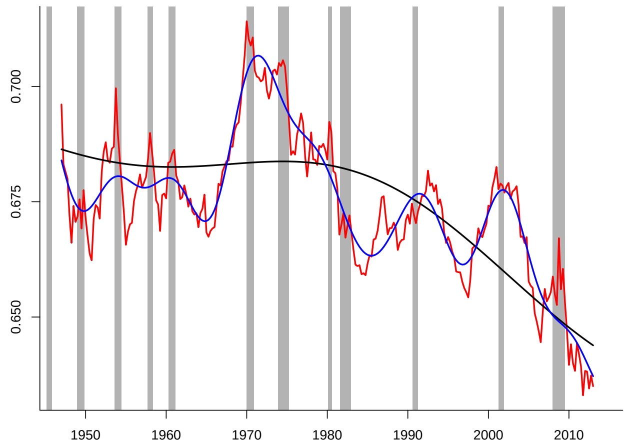
Aggregate hours of skilled workers ($hours_t^U$) - the aggregated hours in all sectors for all employees with education level equivalent at least some college, raw data taken from World KLEMS database.

The ratio of skilled to unskilled hours ($hours_t^S/hours_t^U$) - calculated as a ratio of series described above.

Skill premium (w_t^S/w_t^U) – calculated in the following way. First, we calculate total compensation of employees for both skilled and unskilled workers. The disaggregation is determined by the education achievement as above. Second, we calculate unit compensation per hour for each group. That constructed unit wages are used in final calculation.

B Additional empirical results

Figure B.1: The quarterly labor share and its medium-term components.



Note: the red, blue and black lines represents for raw, medium-term component and long-run trend, respectively.

Figure B.2: The output-to-capital ratio.

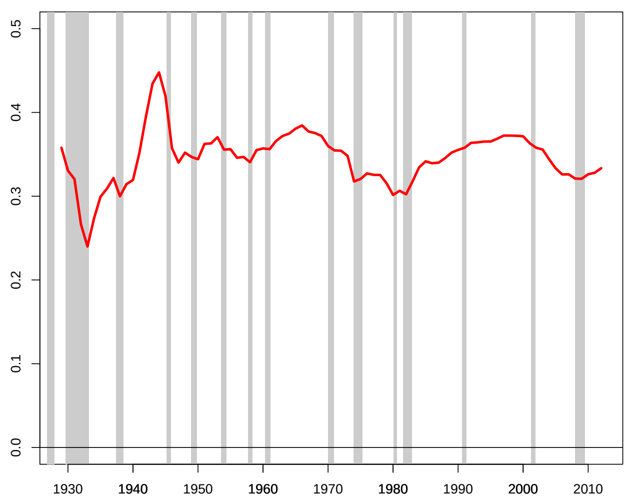
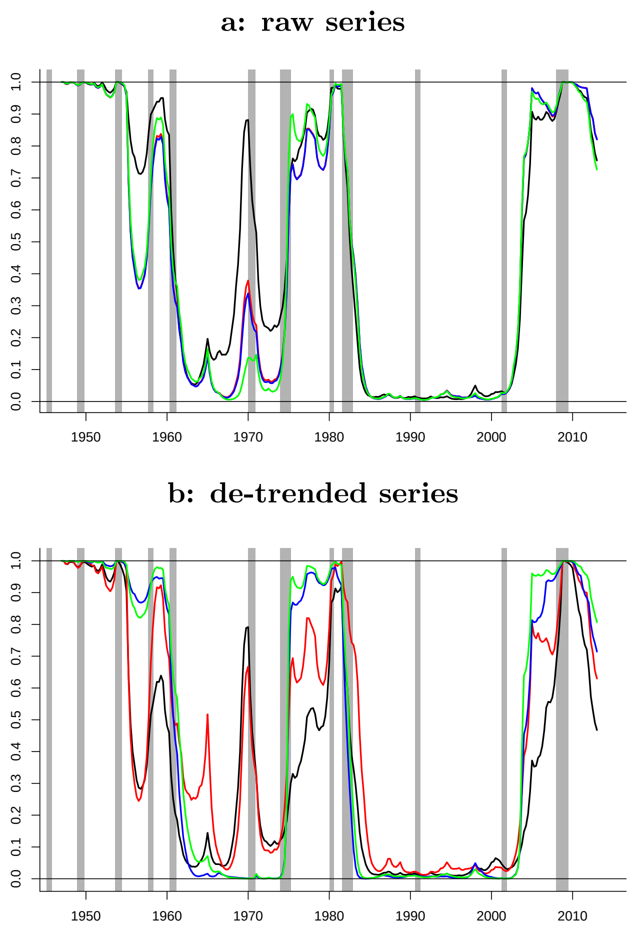


Figure B.3: Filtered probability of the high-variance regime in the Markov switching model.



Note: the black, red, blue and green lines represents smoothed probability of high-variance regime for the specifications (1), (2), (3) and (4), respectively.

Table B.1: Labor share: summary statistics.

	Annual	Quarterly
Mean	0.674	0.674
Max	0.711	0.714
Min	0.621	0.633
Std. Dev.	0.017	0.015
Skewness	−0.369	−0.106
Kurtosis	3.947	3.358
Normality	[0.080]	[0.385]
Obs.	84	265

Note: Normality test is Jarque-Bera.

Table B.2: Labor shares: stationarity tests.

	annual				quarterly			
	intercept		trend & intercept		intercept		trend & intercept	
		CV 5%		CV 5%		CV 5%		CV 5%
ADF	[0.004]	–	[0.004]	–	[0.493]	–	[0.493]	–
ERS DF-GLS	−1.480	−1.940	−2.066	−3.088	−0.353	−1.942	−2.24596	−2.9172
PP	[0.036]	–	[0.022]	–	[0.428]	–	[0.390]	–
KPSS	0.339	0.463	0.245	0.146	0.931	0.463	0.247	0.146
ERS	7.530	3.070	15.913	5.666	12.790	3.199	9.076	5.647
Ng-Perron								
MZa	−4.546	−8.100	−10.444	−17.300	−0.985	−8.100	−10.506	−17.300
MZb	−1.507	−1.980	−2.145	−2.910	−0.346	−1.980	−2.204	−2.910
MSB	0.331	0.233	0.205	0.168	0.351	0.233	0.210	0.168
MPT	5.391	3.170	9.392	5.480	11.312	3.170	9.107	5.480

Note: Regressions performed in levels with sequentially an intercept, then intercept plus linear trend. Squared brackets indicate probability values and CV5% denotes 5% critical value of the test. ADF-Augmented Dickey Fuller; ERS DF-GLS=Elliott-Rothenberg-Stock (1996), Dickey-Fuller GLS; PP=Philips-Perron; KPSS=Kwiatkowski-Phillips-Schmidt-Shin (1992); ERS=Elliott-Rothenberg-Stock (1996) point-optimal unit root; multiple Ng-Perron (2001) tests. Descriptions of these tests can be found widely in contemporary econometrics textbooks. The null in each case is that the series has a unit root. This is except for the KPSS test which has stationarity as the null. In each case the number of lags in the stationarity equation is determined by Schwartz Information criteria. In the Philips-Perron and KPSS methods, we use the Bartlett Kernel as the spectral estimation method and Newey-West bandwidth selection.

C Additional numerical results

Figure C.4: Additional bifurcation figures.

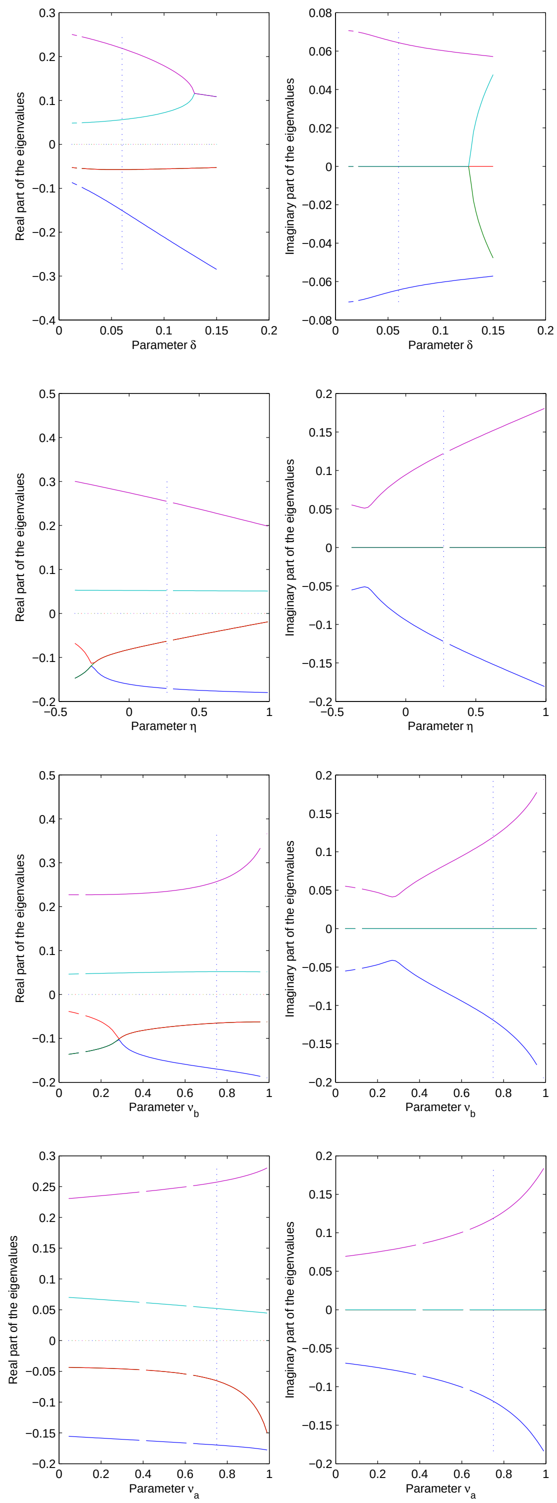


Figure C.5: Non-BGP-preserving bifurcation figures: DC vs. SP.

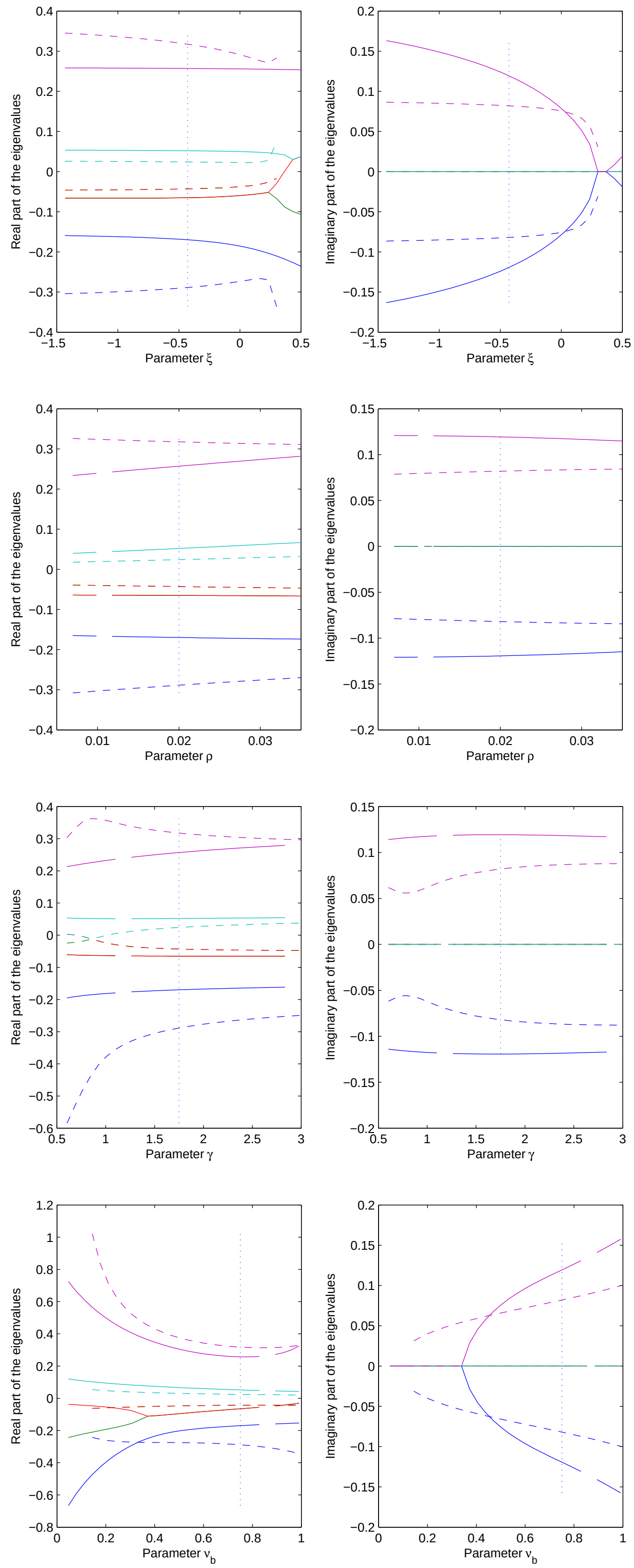


Figure C.6: Non-BGP-preserving bifurcation figures: DC vs. SP (Continued).

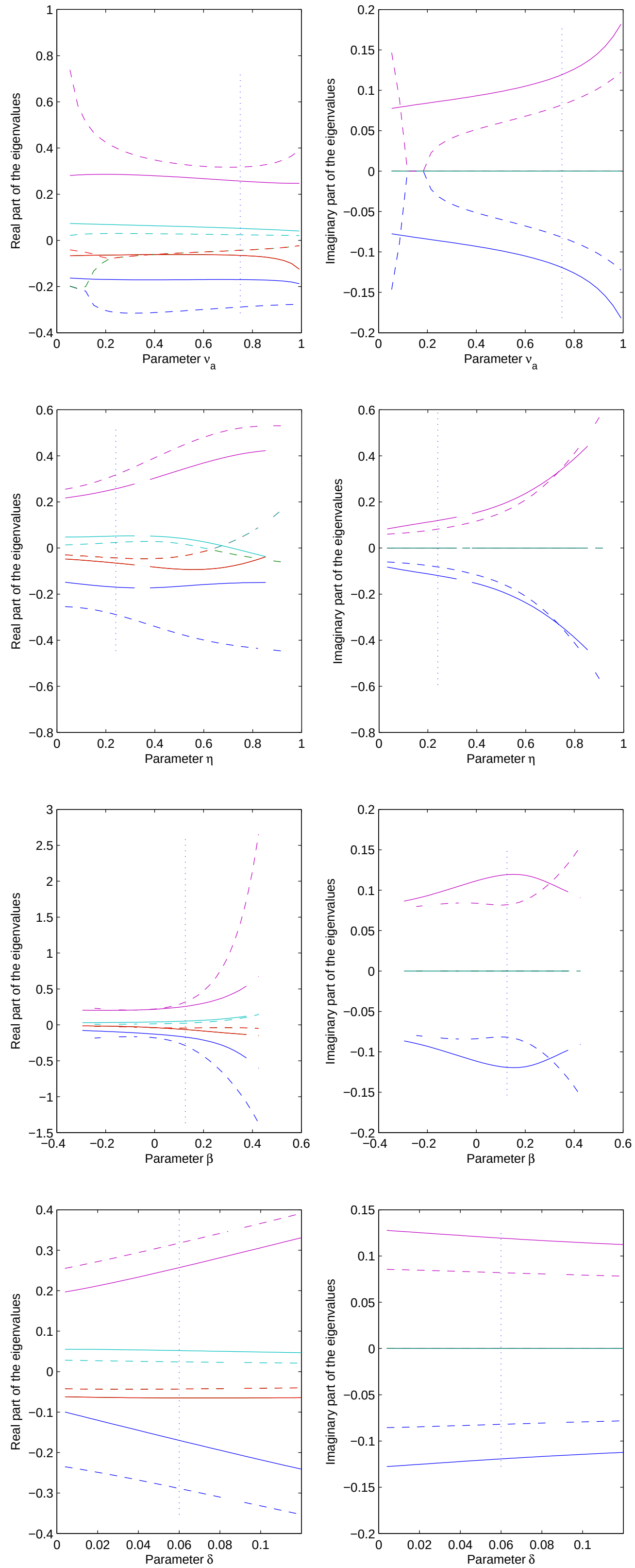


Figure C.7: Incidence of hopf bifurcations for an alternative parametrization, $\nu_a = 0.5, \nu_b = 0.9$.

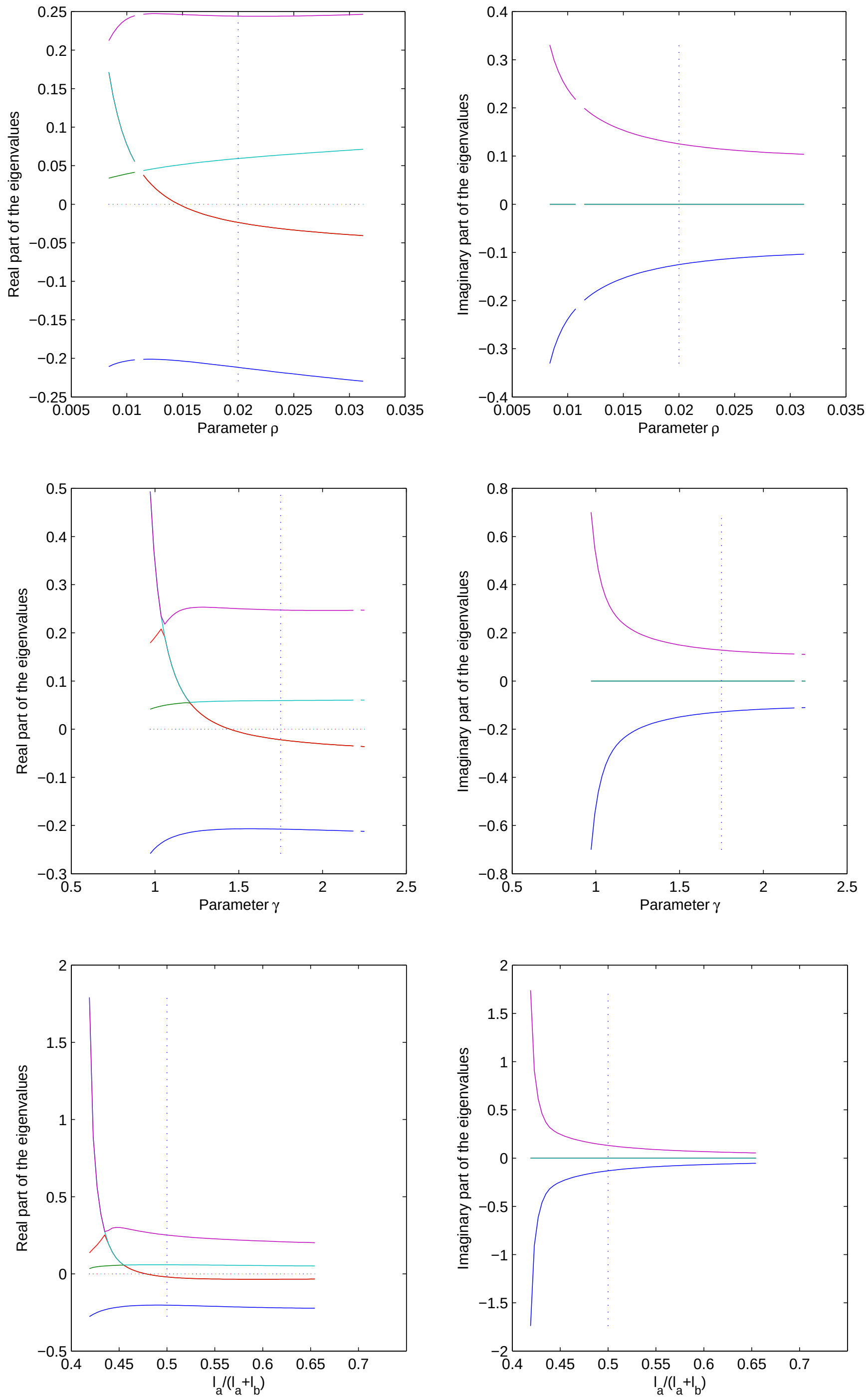


Figure C.8: Comparing balanced growth paths, DC vs. SP. Dependence on the elasticity of substitution.

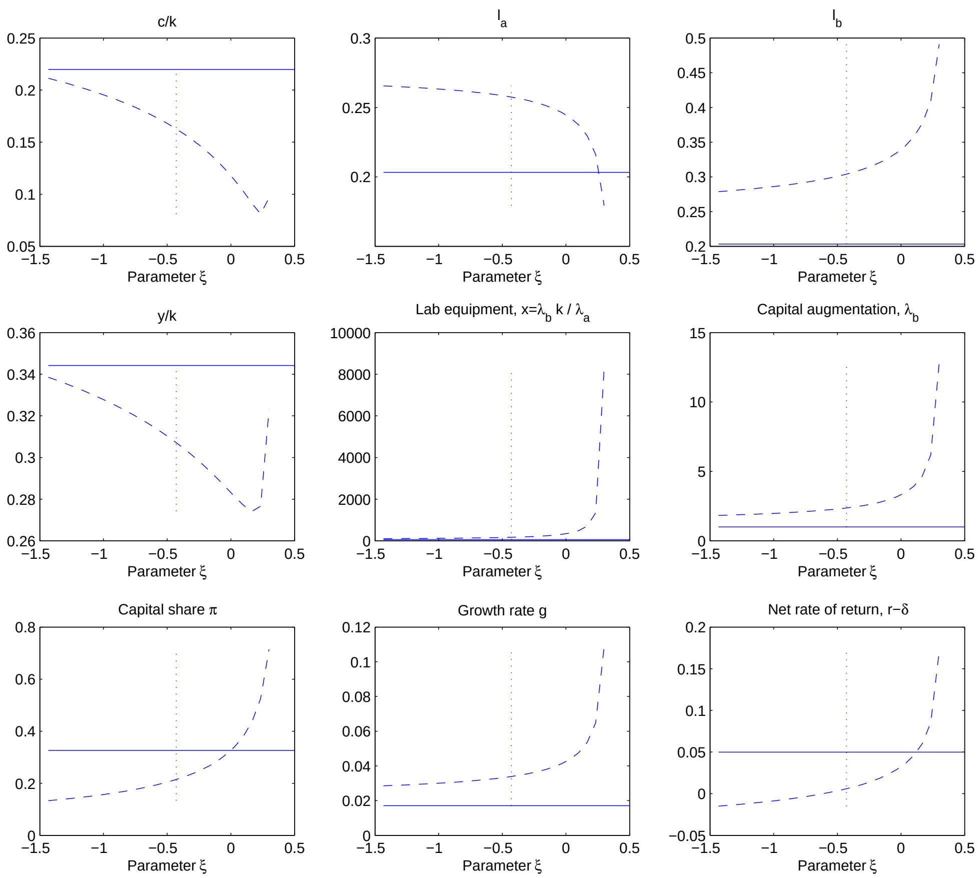


Figure C.9: Comparing balanced growth paths, DC vs. SP. Dependence on the time preference.

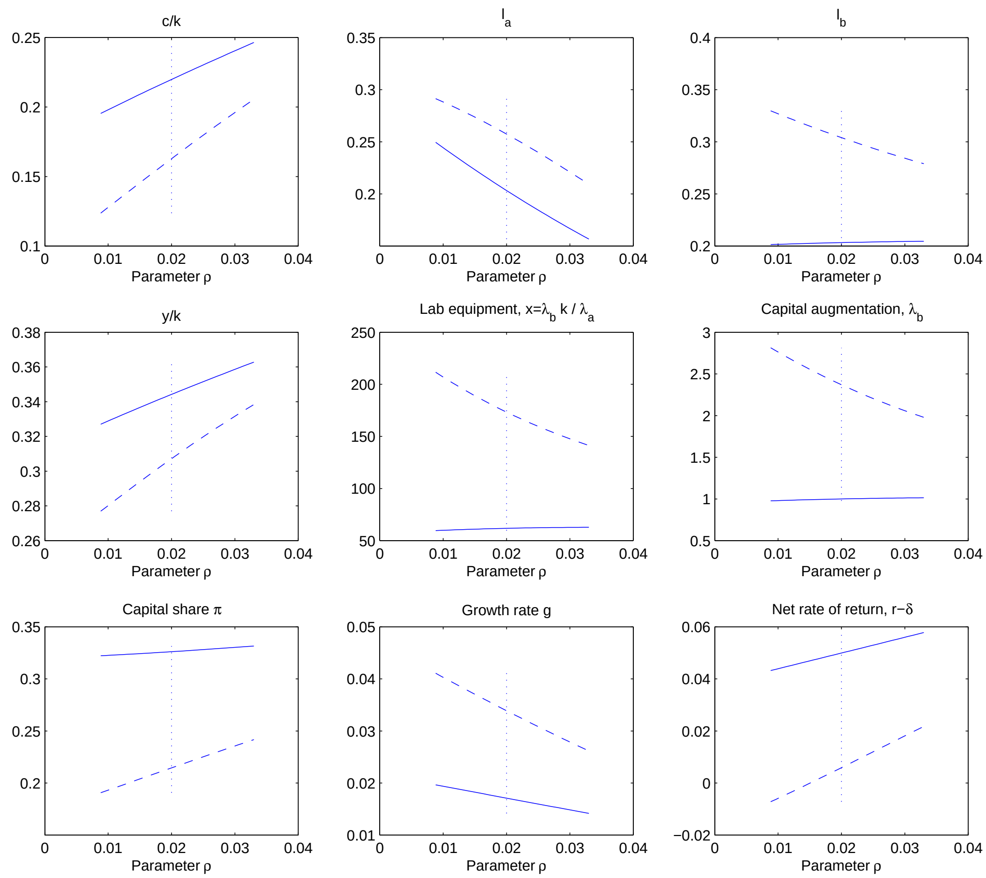


Figure C.10: Comparing balanced growth paths, DC vs. SP. Dependence on the intertemporal elasticity of substitution in consumption.

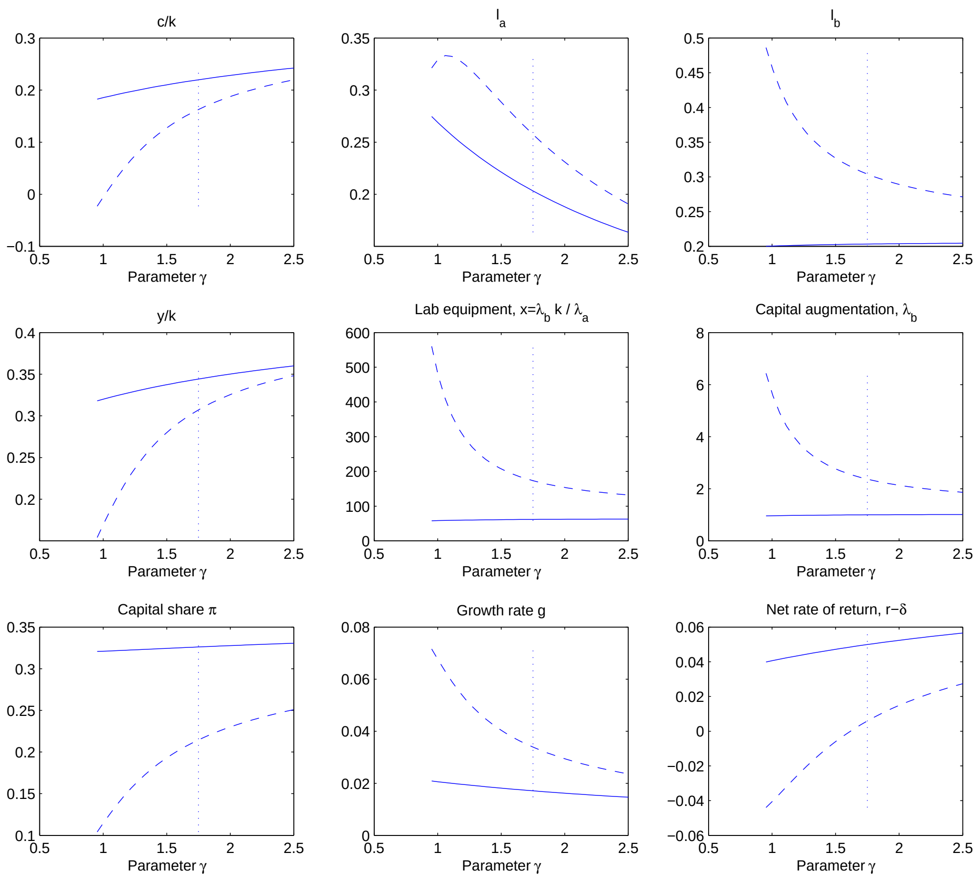


Figure C.11: Comparing balanced growth paths, DC vs. SP. Dependence on ν_b.

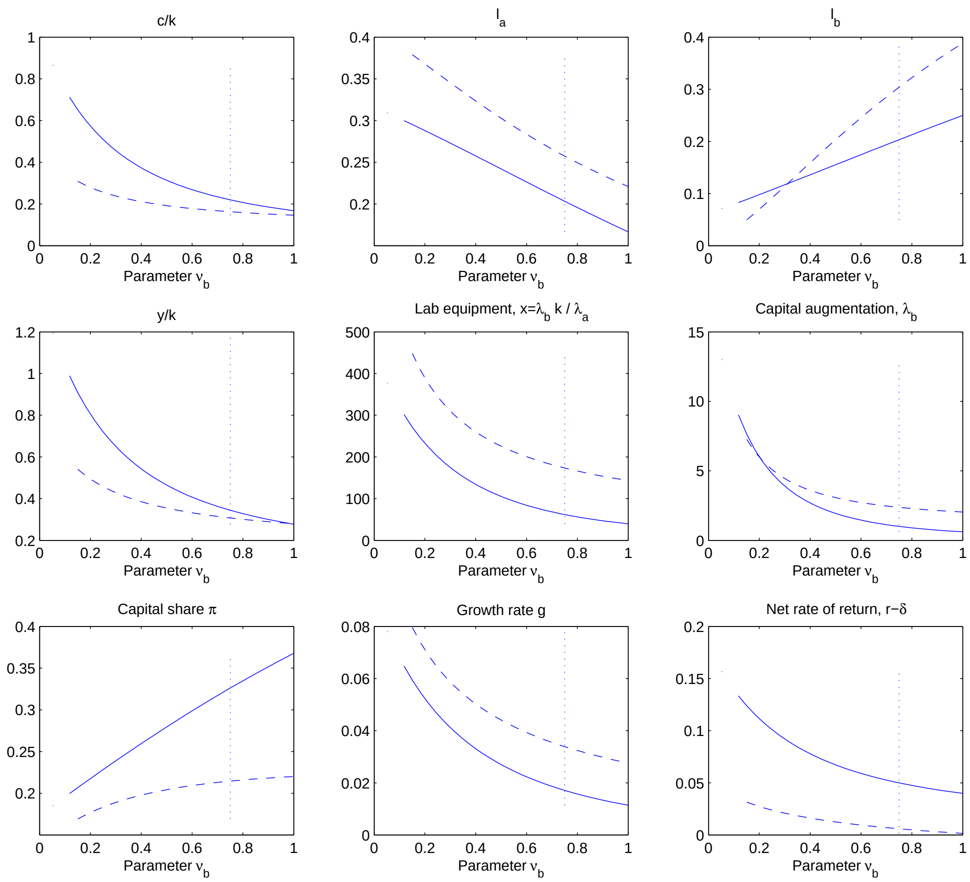


Table C.3: Consequences for varying the baseline calibration for oscillatory dynamics and Hopf bifurcations.

Core calibration with ...	Type of Convergence	Pace of Convergence (per annum)		Cycle Length (in years)		Conditional on ν_a, ν_b, the bifurcation point for		Labor share at BGP
		DC	SP	DC	SP	γ	ρ	(SP alloc.)
$\nu_a = \nu_b = 0.1$	monotonic	0.0484	0.0354	—	—	0.9237	0.0059	0.7230
$\nu_a = \nu_b = 0.5$	dampened	0.0558	0.0430	97.6280	153.7304	0.9534	0.0064	0.7635
$\nu_a = \nu_b = 0.75$	dampened	0.0630	0.0420	52.6563	76.6629	0.9430	0.0062	0.7854
$\nu_a = \nu_b = 0.9$	dampened	0.0944	0.0245	29.4544	42.3207	No Hopf	No Hopf	0.7944
$\nu_a = 0.5, \nu_b = 0.9$	dampened	0.0453	0.0570	54.5507	105.0562	1.0794	0.0086	0.8249
$\nu_a = 0.1, \nu_b = 0.9$	DC - dampened, SP - monotonic	0.0405	0.0357	72.3116	—	1.0864	0.0090	0.8425
$\nu_a = 0.9, \nu_b = 0.1$	DC - monotonic, SP - dampened	0.0432	0.0388	—	161.5625	No Hopf	No Hopf	0.6855

Note: Labor share at BGP in the DC allocation is exactly matched to the long-run US average (0.6739) for each parametrization.