Speculative Trade under Ambiguity *

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Abstract: Ambiguous beliefs may lead to speculative trade and speculative bubbles. We demonstrate this by showing that the classical Harrison and Kreps (1978) example of speculative trade among agents with heterogeneous beliefs can be replicated with agents having common ambiguous beliefs. More precisely, we show that the same asset prices and pattern of trade can be obtained in equilibrium with agents' having recursive multiple-prior expected utilities with common set of probabilities.

While learning about the true distribution of asset dividends makes speculative bubbles vanish in the long run under heterogeneous beliefs, it may not do so under common ambiguous beliefs. Ambiguity need not disappear with learning over time, and speculative bubbles may persist forever.

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1. Introduction.

Ambiguous beliefs may lead to speculative trade and speculative bubbles. We demonstrate this by showing that the classical Harrison and Kreps (1978) example of speculative trade among agents with heterogeneous beliefs can be replicated with agents having common ambiguous beliefs. More precisely, we show that the same asset prices and speculative pattern of trade can be obtained in equilibrium with agents' having multiple-prior expected utilities with common set of probabilities.

The key question of Harrison and Kreps (1978) was whether equilibrium prices in asset markets can persistently exceed all agents' valuations of the asset where valuation is defined by what an agent would be willing to pay if obliged to hold the asset forever. If price exceeds all valuations, then agents who buy the asset must intend to sell it in the future. They trade for short-term gain and hence engage in speculative trade.

Harrison and Kreps considered a model of infinite-time asset markets where risk-neutral agents have heterogeneous expectations about the asset payoffs and cannot short sell the asset. In their model, equilibrium asset price p_t at date t must satisfy the relationship

$$p_t = \max_i \beta E_t^i[p_{t+1} + x_{t+1}], \tag{1}$$

where x_{t+1} denotes dividend at date $t+1$, E_t^i stand for date-t conditional expectation under i'th agent one-period-ahead belief, and β is a discount factor. If beliefs are such that there is no single agent who is more optimistic at all dates and states than other agents about the asset's high dividend next period, then there is no single agent whose expectation is the maximizing one in (1) at all dates and states. This property of one-period-ahead beliefs is called perpetual switching, see Morris (1996). Under perpetual switching, asset prices (strictly) exceed every agent's valuation. There is speculative bubble. There is also persistent speculative trade because the agent whose belief is the maximizing one in (1) holds the asset at date t while agents whose beliefs give strictly lower expectations sell the asset if they had some holdings from previous date. A model of asset markets in continuous time exhibiting the same features has been studied by Scheinkman and Xiong (2003). Heterogeneous beliefs in their model are generated by agents' overconfidence in informativeness some some signals about future dividends. As signals arrive over time, agents' beliefs about future dividends exhibit switching. This gives rise to speculative bubbles.

We consider the same model of asset markets as Harrison and Kreps (1978) except for the specification of agents' beliefs. Instead of heterogeneous but exact beliefs, agents in our model have common ambiguous beliefs described by sets of one-period-ahead probabilities. Their decision criterion is the recursive multiple-prior expected utility - an extension of Gilboa and Schmeidler (1989) maxmin criterion to dynamic setting due to Epstein and Schneider (2003). Our key observation is that equilibrium pricing relationship (1) continues to hold with expectation E_t^i of agent i being now the one-period-ahead belief that minimizes expected value of that agent's date-$(t+1)$ consumption (more precisely, date-$(t+1)$ continuation utility) over the set of multiple probabilities. We call those probabilities effective one-period-ahead beliefs, and they feature in the valuation of agents' willingness to pay for the asset if obliged to hold it forever. If there is sufficient heterogeneity of agents' equilibrium consumption plans, then effective beliefs have switching property and - as in Harrison and Kreps (1978) - equilibrium prices exceed all agents' asset valuations. Further, there is speculative trade. Heterogeneity of equilibrium consumption is generated by heterogeneous initial endowments. Initial endowments play a critical role under common ambiguous beliefs, in contrast to the case of heterogeneous beliefs where they do not matter.

Two objections have been frequently raised against Harrison and Kreps' model of speculative trade. The first is that it departs from the common-prior assumption. This objection does not apply to our model. Agents in our model have common priors, or more precisely, common set of priors. The second is that agents have dogmatic beliefs and do not learn from observations of realized dividends over time. In response to this objection, Morris (1996) introduced learning in the model of speculative trade. He considered an i.i.d dividend process parametrized by a single parameter of its distribution (probability of high dividend) that is unknown to the agents. Agents have heterogeneous prior beliefs about that parameter. Morris (1996) showed that, as the agents update their beliefs over time, their posterior beliefs will exhibit switching property that leads to speculative trade. It is a standard result in Bayesian learning (Blackwell and Dubins' (1962) merging of opinions) that agents posterior beliefs converge to the true parameter of the dividend distribution. This implies that asset prices converge to agents' valuations as time goes to infinity.

A slightly different model of speculative trade under heterogeneous beliefs with learning has been studied in Slawski (2008). In his model the dividend process is a Markov chain. Agents don't know the true transition matrix and have heterogeneous priors on a set of possible transition matrices. Slawski (2008) provides conditions under which there is speculative trade in equilibrium when agents update their beliefs. Under fairly general conditions, posterior beliefs converge to true parameter as time goes to infinity, and asset prices converge to agents' valuations.

Learning and updating of beliefs can be significantly different under ambiguity than with no ambiguity.[1] Depending on the interaction between random experiments that generate dividends ambiguity about dividends may or may not fade away in the long run. The dividend process in our version of Harrison and Kreps' model can be thought of as resulting from sequences of indistinguishable but unrelated experiments, see Epstein and Schneider (2003b). Such experiments give rise to persistent ambiguity that is unaffected by learning. Morris' (1996) model of speculative trade under heterogeneous beliefs with learning can be replicated with common ambiguous beliefs. Agents have common set of prior beliefs about a parameter of probability distribution of an i.i.d process of dividends. They update their beliefs prior-by-prior in Bayesian way. If agents' consumption plans exhibit sufficient heterogeneity, effective posterior beliefs have the switching property and there results speculative trade. The critical condition is heterogeneity of agents' initial endowments. In this case, ambiguity about dividends fades away in the long run because posterior beliefs converge to true probability. As in Morris (1996) asset prices converge to valuations and the speculative bubble converges to zero.

In their comprehensive study of the dot.com bubble of 2000-2001 Ofek and Richardson (2003) concluded that short-sales restrictions and heterogeneity of investors' beliefs were the main reasons for the dramatic rise and fall of prices of internet stocks during that period. Short sales restrictions on internet stocks were particularly stringent because of the so-called lockups. Their main argument in support of belief heterogeneity was relatively low level of institutional holdings of

[1]Epstein and Schneider (2007) propose a model of learning with multiple priors that may leave some ambiguity remaining in the long run.

internet stocks. Individual investors tend to have more diverse beliefs. Yet, it is hard to believe that investors could have so diverse beliefs over a relatively long period of time. The argument of merging of opinions implies that diverse beliefs should quickly disappear. Our findings offer a different interpretation of Ofek and Richardson's analysis. Instead of being diverse, investors beliefs could have been ambiguous but common. Majority of dot.com stocks were new to the market justifying potential ambiguity of investors beliefs. Ambiguity of beliefs could persist over a long period of time.

The results of this paper are in stark contrast to the existing literature on implications of ambiguous beliefs (or ambiguity aversion) on equilibrium in asset markets. Inspired by the portfolio-inertia result of Dow and Werlang (1992), the literature has strived to demonstrate that non-participation in trade by some agents with ambiguous beliefs may arise in equilibrium. In Cao, Wang and Zhang (2005) agents have heterogeneous ambiguity and those with the highest degree of ambiguity opt out of trading risky assets in equilibrium.[2] Mukerji and Tallon (2001) show that ambiguous beliefs concerning idiosyncratic risk may lead to break down of trade of some assets.

The paper is organized as follows. In Section 2 we review the model of Harrison and Kreps (1978) of speculative trade under heterogeneous beliefs. In Section 3 we show how the same asset prices and asset holdings can be obtained in equilibrium with agents having recursive multiple-prior expected utilities with common set of probabilities. In Section 4 we extend the model and identify the property of switching beliefs that is shown to give rise to speculative bubbles. A model of speculative trade with learning under ambiguity is presented in Section 5.

2. Speculation under Heterogeneous Beliefs.

The following example is due to Harrison and Kreps (1978). There is a single infinitely-lived asset with uncertain dividend equal to either 0 (low dividend) or 1 (high dividend) at every date $t \geq 1$. Date-0 dividend is equal to zero. Let $S = \{0, 1\}$ be the set of states for every date. There are two agents who perceive

[2]Further studies of limited participation in trade under ambiguity are Easley and O'Hara (2009), Illeditsch (2011), and Ozsoylev and Werner (2011).

the dividend process $\{x_t\}$ as Markov chain with different transition probabilities, that is, they have *heterogeneous beliefs*. Transition probabilities are specified by probabilities of next-period high dividend conditional on current state, $q^i(s)$, with the respective probability of low dividend being $1 - q^i(s)$. Suppose that

$$q^1(0) = \frac{1}{2} \qquad q^1(1) = \frac{1}{3}, \tag{2}$$

for agent 1, and

$$q^2(0) = \frac{1}{3}, \qquad q^2(1) = \frac{3}{4} \tag{3}$$

for agent 2. The key feature of transition probabilities (2 - 3) is the property of switching beliefs: Agent 1 is more optimistic than agent 2 about next-period high dividend when current dividend is zero, while it is the other way round when current dividend is one.

Agents are risk-neutral with utility functions over infinite-time consumption plans given by the discounted expected value

$$u^i(c) = E^i[\sum_{t=0}^{\infty} \beta^t c_t], \tag{4}$$

where E^i denotes expectation under the unique probability measure on S^∞ derived from transition probabilities q^i. The common discount factor is $\beta = 0.75$. Consumption endowments don't matter and are left unspecified. The asset supply is normalized to one share which is initially held by agent 1. Short selling of the asset is prohibited in that there is zero short-sales constraint.

In equilibrium, the agent who is more optimistic at any date and state holds the asset and the price reflects his one-period valuation of the payoff. The less optimistic agent wants to sell the asset short and ends up with zero holding because of the short-sales constraint. There exists a stationary equilibrium with asset prices that depend only on the current dividend. Equilibrium prices, denoted by $p(0)$ and $p(1)$, obtain from the first-order conditions for the respective optimistic agent,

$$p(0) = \beta[(1 - q^1(0))p(0) + q^1(0)(p(1) + 1)] \tag{5}$$

$$p(1) = \beta[(1 - q^2(1))p(0) + q^2(1)(p(1) + 1)] \tag{6}$$

They are

$$p(0) = \frac{24}{13}, \qquad p(1) = \frac{27}{13}. \tag{7}$$

Security holdings are

$$h^1(0) = 1, \quad h^1(1) = 0, \qquad h^2(0) = 0, \quad h^2(1) = 1. \tag{8}$$

The first-order conditions for agent 2 when the dividend is zero and for agent 1 when the dividend is one hold as strict inequalities. Transversality conditions hold, too.

The discounted expected value of the asset's future dividends at date t under agent's i beliefs is

$$V_t^i(s) = \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} E^i[x_\tau | x_t = s] \tag{9}$$

for $s = 0, 1$. The value $V_t^i(s)$ does not depend on t and we drop subscript t from the notation. Because of linearity of utility functions, discounted expected value of dividends is the agent's willingness to pay for the asset if obliged to hold forever. Therefore $V^i(s)$ may be called the *fundamental value*.[3] If asset price $p(s)$ strictly exceeds every agent's fundamental value, then there is *speculative bubble*. An agent who buys the asset at price $p(s)$ must be planning to sell it at a future date. Thus, the agent engages in speculative trade. The difference between the price and the maximum of fundamental values is termed *speculative premium*.

Elementary algebra shows that

$$V^1(0) = \frac{4}{3}, \qquad V^1(1) = \frac{11}{9}, \tag{10}$$

$$V^2(0) = \frac{16}{11}, \qquad V^2(1) = \frac{21}{11}. \tag{11}$$

It holds

$$p(0) > V^i(0) \text{ and } p(1) > V^i(1)$$

for $i = 1, 2$. There is speculative bubble in every state at every date.

3. Speculation under Ambiguous Beliefs.

Consider the same asset as in Section 2 with dividends equal to 0 or 1 at every date, and no short selling. The specification of agents' endowment is important

[3] Needless to say, this is a different notion of fundamental value than the one used in the literature on rational price bubbles, see for example Werner (2012).

here. Suppose that the endowment e_t^1 of agent 1 can take two possible values 10 and 5 for every date $t \geq 1$. Agent's 2 endowment is $e_t^2 = 15 - e_t^1$. The joint process (x_t, e_t^1) takes four possible values $(0,5), (0,10), (1,5), (1,10)$ for every date $t \geq 1$. Those four values constitute the state space Ω at every date. Date-0 state is $(0,10)$.

Agents have common ambiguous beliefs that are described by sets of transition (or one-period-ahead) probabilities. These sets are constructed using the probabilities from Section 2 in the following way. For states $(0,5)$ and $(0,10)$, that is, when the current dividend is zero, the set of transition probabilities is the convex hull of two probability vectors $((1-q^1(0)), 0, q^1(0), 0)$ and $(0, (1-q^2(0)), 0, q^2(0))$, where $q^1(0)$ and $q^2(0)$ are as in eqs. (2 - 3). Thus

$$\mathcal{P}(0) = co\ \{(\frac{1}{2}, 0, \frac{1}{2}, 0),\ (0, \frac{2}{3}, 0, \frac{1}{3})\}. \tag{12}$$

Equivalently, $\mathcal{P}(0)$ can be described as the set of probability vectors $\big(1/2\lambda,\ 2/3(1-\lambda), 1/2\lambda,\ 1/3(1-\lambda)\big)$ over all $\lambda \in [0,1]$. Note that under the specification (12) the upper probability of high dividend next period is 1/2 or $q^1(0)$ while the lower probability is 1/3 or $q^2(0)$.[4] Upper and lower probabilities of high endowment (equal to 10) next period are 1 and 0, respectively. Thus there is maximal ambiguity about high or low endowment. For an example of an experiment with two Ellsberg urns that leads to a set of probabilities of the form (16), see Couso et al (1999), Example 5.

The set of transition probabilities for the event of dividend equal to one obtains from probabilities $q^1(1)$ and $q^2(1)$ in an analogous way. It is

$$\mathcal{P}(1) = co\ \{(2/3, 0, 1/3, 0),\ (0, 1/4, 0, 3/4)\}. \tag{13}$$

Here, the upper probability of high dividend next period is 3/4 or $q^2(1)$ while the lower probability is 1/3 or $q^1(1)$.

Agents have identical recursive multiple-prior expected utilities, with linear period utility, defined by

$$u_t(c, s) = c_t(s) + \beta \min_{P \in \mathcal{P}(s)} E_P[u_{t+1}(c)], \tag{14}$$

[4]Upper probability of an event $A \subset \Omega$ for a set of probability measures $\mathcal{P}$ on Ω is defined as $\max_{P \in \mathcal{P}} P(A)$. Lower probability of A is the minimum over the same set.

where $u_t(\cdot, s)$ is date-t utility function in state s. The discount factor is $\beta = 0.75$. Date-0 utility function implied by this recursive relation is

$$u_0(c) = \min_{\pi \in \Pi} E_\pi[\sum_{t=0}^{\infty} \beta^t c_t], \tag{15}$$

where Π is a set of probabilities on S^∞ such that conditional one-period-ahead probabilities at any date t in state s are $\mathcal{P}(s)$, see Epstein and Schneider (2003).

We claim that asset prices (7) and asset holdings (8) derived in Section 2 are an equilibrium. To see this consider consumption plan of agent 1 resulting from asset holdings (8) at prices (7). For a date-t event where the asset pays zero dividend, transition probabilities that minimize the expected value of agent's 1 next period continuation utility over all probabilities in $\mathcal{P}(0)$ are $(\frac{1}{2}, 0, \frac{1}{2}, 0)$ (see Appendix). Price $p(0)$ equals the discounted expected one-period payoff under these probabilities (see eq. (6)). Therefore holding one share of the asset is an optimal choice of agent 1. Transition probabilities that minimize the expected value of agent's 2 next period continuation utility of her consumption plan resulting from asset holdings (8) at prices (7) over the set $\mathcal{P}(0)$ are $(0, \frac{2}{3}, 0, \frac{1}{3})$. Under those probabilities the discounted expected one-period payoff of the asset is strictly less than the price $p(0)$. As explained in the Appendix, the optimal holding of agent 2 is zero.

Consider next the event where the asset pays high dividend. Here, the probabilities that minimize the expected value of agent's 2 next period continuation utility over the set $\mathcal{P}(1)$ are $(0, \frac{1}{4}, 0, \frac{3}{4})$. Since price $p(1)$ equals the discounted expected payoff under these probabilities (see again eq. (6)), holding one share of the asset is an optimal choice for agent 2. Probabilities that minimize the expected value of agent's 1 continuation utility over $\mathcal{P}(1)$ are $(\frac{2}{3}, 0, \frac{1}{3}, 0)$. The discounted expected payoff of the asset under these probabilities is strictly less than the price $p(1)$. The optimal holding of agent 1 is zero.

Figure 1 in the Appendix provides graphical intuition (albeit with two states) for the type of equilibrium we obtained. It makes clear that heterogeneity of agents endowments, which appears here in a strong form of negative comonotonicity between e_t^1 and e_t^2, is crucial for speculative trade under ambiguity. The equilibrium price $p(0)$ is equal to the maximum discounted expected one-period payoff over all probabilities in the set $\mathcal{P}(0)$. The same holds for price $p(1)$ and the set $\mathcal{P}(1)$.

An agent's willingness to pay for the asset if obliged to hold forever is the discounted expected value of future dividends under probabilities that minimize the expected value of the agent's next period continuation utility. Therefore, fundamental values of the asset remain the same $V^i(0)$ and $V^i(1)$ for $i = 1, 2$ as in eq. (11). Equilibrium prices $p(0)$ and $p(1)$ strictly exceed both agents' fundamental valuations and there is speculative bubble. Speculative premium is time independent.

4. Belief Switching and Speculation.

In this section we consider a generalization of the model of Section 3 that will be useful for the analysis of learning in Section 5. We maintain the assumptions that there are two agents with endowments as in Section 3 and that the dividend takes two values 0 or 1. Let s_t denote a history of dividends from date 1 through date t, for $t \geq 1$. We consider sets of one-period-ahead probabilities $\mathcal{P}_t(s_t)$ in event s_t that are convex hulls of two probability vectors on Ω, $Q^1_t(s_t) = ((1 - q^1_t(s_t)), 0, q^1_t(s_t), 0)$ and $Q^2_t(s_t) = (0, (1 - q^2_t(s_t)), 0, q^2_t(s_t))$, that depend on the history of dividends but not on agents' endowments. Thus

$$\mathcal{P}_t(s_t) = co\ \{Q^1_t(s_t),\ Q^2_t(s_t)\}. \tag{16}$$

Equivalently, $\mathcal{P}_t(s_t)$ can be described as $\big(\lambda(1-q^1_t(s_t)),\ (1-\lambda)(1-q^2_t(s_t)), \lambda q^1_t(s_t),\ (1-\lambda)q^2_t(s_t)\big)$ for all $\lambda \in [0, 1]$. Note that the upper probability of high dividend next period is $\max\{q^1_t(s_t), q^2_t(s_t)$ while the lower probability is $\min\{q^1_t(s_t), q^2_t(s_t)\}$. The upper and lower probabilities of high endowment next period are 1 and 0, respectively. Thus there is maximal ambiguity about high or low endowment.[5] In the example of Section 3, one-period-ahead probabilities depend only on current dividends and are obtained from transition probabilities $q^i(s)$ given by (2 - 3).

Agents have identical recursive multiple-prior expected utilities with linear period utility specified by

$$u_t(c, s_t) = c_t(s_t) + \beta \min_{P \in \mathcal{P}(s_t)} E_P[u_{t+1}(c)|s_t], \tag{17}$$

[5]This extreme ambiguity is not essential for our results, but the presence of some ambiguity about endowments is essential.

with time- and event-dependent sets of one-period-ahead probability measures $\mathcal{P}_t(s_t)$.

Consider an equilibrium consisting of asset prices p and allocations (c^1, c^2) of consumption and (h^1, h^2) of asset holdings. Because utility functions are linear in current consumption and one-period-ahead probabilities depend only on dividend history, we postulate that equilibrium prices depend only on dividend histories. Let $P^i_t(s_t)$ be the probability in $\mathcal{P}_t(s_t)$ that minimizes the expected value of next-period continuation utility of agent's i equilibrium consumption. That is,

$$P^i_t(s_t) = \text{argmin}_{P \in \mathcal{P}_t(s_t)} E_P[u_{t+1}(c^i)|s_t]. \tag{18}$$

We assume that $P^i_t(s_t)$ is unique.[6] We call probability $P^i_t(s_t)$ the *effective belief* of agent i at consumption plan c^i.

Equilibrium prices must satisfy

$$p_t(s_t) = \beta \max_i E_{P^i_t(s_t)}[p_{t+1} + x_{t+1}|s_t], \tag{19}$$

for every s_t. The agent whose effective belief is the maximizing one in (19) holds the asset in event s_t while the other agent whose beliefs give strictly lower expectation has zero holding (see Appendix).

Let $\hat{P}_t(s_t)$ denote the maximizing probability on the right-hand side of (19). Further, let π^i for $i = 1, 2$ and $\hat{\pi}$ be probability measures on Ω^∞ derived from one-period-ahead probabilities P^i_t and $\hat{P}_t$, respectively. Note that $\hat{\pi}$ is the risk-neutral pricing measure (or state-price process) for p. We apply Theorem 3.3 of Santos and Woodford (1997) to conclude that equilibrium price of the asset must be equal to the infinite sum of discounted expected dividends under the risk-neutral measure. That is,

$$p_t(s_t) = \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} E_{\hat{\pi}}[x_\tau|s_t], \tag{20}$$

for every s_t. The fundamental value of the asset is the sum of discounted expected dividends under the agent's effective beliefs. That is,

$$V^i_t(s_t) = \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} E_{\pi^i}[x_\tau|s_t], \tag{21}$$

[6] We make this assumption to simplify exposition. In all examples discussed in this paper the uniqueness condition is satisfied. See Appendix for further discussion.

for every s_t. It follows from eq. (19) that $p_t(s_t) \geq V_t^i(s_t)$ for every i.

Following Morris (1996), we say that transition probabilities $\{q_t^1(s_t)\}$ and $\{q_t^2(s_t)\}$ exhibit *perpetual switching*, if for every $i = 1, 2$ and state s_t there exists $\tau > t$ and state s_τ which is a successor of s_t such that $q_t^i(s_\tau) < \hat{q}_t(s_\tau)$, where $\hat{q}_t(s_t) = \max\{q_t^1(s_t), q_t^2(s_t)\}$.

We have the following:

Theorem 1: *Suppose that probabilities $q_t^i(s_t)$ depend only on the number of high dividends from date 1 through t, for $i = 1, 2$. If $\{q_t^1(s_t)\}$ and $\{q_t^2(s_t)\}$ exhibit perpetual switching, then in equilibrium*

$$p_t(s_t) > V_t^i(s_t), \tag{22}$$

for $i = 1, 2$ and for all s_t.

Proof: The proof consist of three steps. First, we show that the effective belief of agent 1 in state s_t is $Q_t^1(s_t)$. That is, $P_t^1(s_t) = Q_t^1(s_t)$. Similarly, $P_t^2(s_t) = Q_t^2(s_t)$. Second, we show that if $\hat{q}_t(s_t) > q_t^i(s_t)$, then $\hat{P}_t(s_t) \neq Q_t^i(s_t)$. That is, the equilibrium asset price (19) in state s_t at date t is equal to the discounted expected one-period payoff under the effective belief of the agent who assigns higher probability of next period high dividend. This implies that holding one share of the asset is optimal for that agent while zero holding is optimal for the other agent guaranteeing that markets clear. Lastly, we show that if there is perpetual switching, then speculative bubble must be strictly positive.

The proof of steps 1 and 2 can be found in Appendix. Here we prove the last step. It follows from eq. (19) that

$$p_t(s_t) \geq \sum_{\tau=t+1}^{T} \beta^{\tau-t} E_{\pi^i}[x_\tau|s_t] + \beta^{T-t} E_{\pi^i}[p_T|s_t], \tag{23}$$

for every i and every s_t and $T > t$. The assumption of perpetual switching implies that there is s_τ, a successor of s_t, such that $\hat{q}_\tau(s_\tau) > q_\tau^i(s_\tau)$. Then $\hat{P}_\tau(s_\tau) \neq Q_\tau^i(s_\tau)$, and for every $T \geq \tau$ inequality in (23) is strict. Since the right-hand side of (23) is non-increasing in T and it converges to $V_t^i(s_t)$ as T goes to infinity, we obtain (22). $\square$

Theorem 1 is a counterpart of Theorem 2 in Morris (1996) for heterogeneous beliefs (see also Proposition 2.8 in Slawski (2008)). Equilibrium asset prices in

Theorem 1 satisfy the recursive relation

$$p_t(s_t) = \beta\big[(1-\hat{q}_t(s_t))p_{t+1}(s_t,0) + \hat{q}_t(s_t)(p_{t+1}(s_t,1)+1)\big], \tag{24}$$

where $(s_t, 0)$ and $(s_t, 1)$ are the two continuation histories of s_t at date $t+1$. Similar recursive relations for fundamental values of the asset are

$$V_t^i(s_t) = \beta\big[(1-q_t^i(s_t))V_{t+1}^i(s_t,0) + q_t^i(s_t)(V_{t+1}^i(s_t,1)+1)\big]. \tag{25}$$

Harrison and Kreps (1978) (see also Slawski (2008)) provide a method of calculating solutions to pricing equation (24).

5. Speculation and Learning.

In the Harrison and Kreps' example of speculative trade under heterogeneous beliefs agents have dogmatic beliefs. They neither update their beliefs nor learn in any way from past observations of dividends. To address this shortcoming Morris (1996) introduced learning in the model of speculative trade with heterogeneous beliefs. He considered an i.i.d dividend process and assumed that the probability of high dividend is unknown to agents. Agents have heterogeneous prior beliefs about the probability of high dividend. Morris (1996) showed that, as the agents update beliefs over time, their posterior beliefs will exhibit perpetual switching for a large class of heterogeneous priors. Thus learning generates speculative trade in the model with i.i.d dividend process and heterogeneous priors. It is a standard result in Bayesian learning (see Blackwell and Dubins (1962)) that conditional posterior beliefs converge to each other over time. This implies that speculative premium under learning converges to zero over time. In contrast, speculative premium in the Harrison and Kreps' example is time independent.

The model of Morris (1996) of speculative trade under heterogeneous beliefs with learning can be replicated with common ambiguous beliefs. It provides an interesting perspective on learning under ambiguity. Suppose that the dividend process x_t is i.i.d taking values 0 and 1. The true probability of high dividend $\pi \in [0, 1]$ is not known to the agents. They have prior beliefs about π. Prior beliefs are probability measures on $[0, 1]$, that is, elements of the set of measures $\mathcal{M}([0, 1])$. Because of ambiguity, agents do not have unique prior but instead they have multiple prior beliefs. More specific, there are two priors $\mu^1 \in \mathcal{M}([0,1])$ and $\mu^2 \in \mathcal{M}([0,1])$ assumed to have density functions (twice differentiable and bounded below) on $[0,1]$. The endowment process e_t^1 taking values 5 and 10 is an I.I.D. process (see Epstein and Schneider (2003b)) with time- and state-independent set of one-period-ahead probabilities $\mathcal{P} = co\{(0,1),(1,0)\}$, that is, $\mathcal{P} = \Delta^2$.

Agents update their prior beliefs about the dividend process using observations of realized dividends. Let $\theta_t^i(s_t)$ be the posterior probability of next-period high dividend conditional on observed history of dividends in event s_t at date t derived from prior μ^i. No updating of beliefs about endowment process takes place since the underlying random experiments at different dates are considered unrelated but indistinguishible. The interaction between processes x_t and e_t^1 is characterized by independence in the selection, see Couso et al (1999). That is, the set of one-period-ahead probabilities for the joint proces (x_t, e_t^1) at date t is

$$\mathcal{P}_t(s_t) = co\ \{(1-\theta_t^1(s_t), 0, \theta_t^1(s_t), 0),\ (0, 1-\theta_t^2(s_t), 0, \theta_t^2(s_t))\}. \tag{26}$$

Agents have recursive multiple-prior expected utilities (17) with sets of one-period-ahead probabilities given by (26).

Morris (1996) shows that under fairly general conditions sequences of posterior probabilities $\{\theta_t^1(s_t)\}$ and $\{\theta_t^2(s_t)\}$ exhibit perpetual switching, that is, for every $i = 1,2$ and event s_t there exists $\tau > t$ and event s_τ which is a successor of s_t such that $\theta_\tau^i(s_\tau) < \hat{\theta}_\tau(s_\tau)$, where $\hat{\theta}_t(s_t) = \max\{\theta_t^1(s_t), \theta_t^2(s_t)\}$. For example, if one prior is the uniform (or "ignorance") prior while the second is Jeffreys' prior, then there is perpetual switching (see Example 1 in Morris (1996)). It follows from Theorem 1 that there is strictly positive speculative premium and speculative trade in equilibrium.

Because prior beliefs about dividends are absolutely continuous with respect to each other, posterior probabilities of high dividend converge to the true probability π. This implies that speculative premium converges to zero as time goes to infinity. In contrast, the marginal probability of high endowment of agent 1 lies between 0 and 1 at every date. Ambiguity about endowments does not fade away. Needless to say, this persistent ambiguity about endowments has no effect on speculative premium in this model.

Learning and updating of beliefs can indeed be significantly different under ambiguity than with no ambiguity. Depending on the interaction between repeated experiments ambiguity may or may not fade away in the long run. In our version of the model of Morris (1996), the dividend process is belived to be i.i.d. and the ambiguity fades away over time with learning (see Marinacci (2002) for more general results along these lines). In contrast, the endowment process has unknown relationship between subsequent experiments and hence is an I.I.D. process with persistent ambiguity. Similarly, the joint dividend-endowment process in our version of Harrison and Kreps' model can be thought of as resulting from two sequences of indistinguishable experiments with unknown relationship. Epstein and Schneider (2007) propose a general model of learning with multiple priors that may leave some ambiguity remaining in the long run.

6. Appendix.

Portfolio choice under ambiguity with risky endowments.

We first explain the features of optimal portfolio choice under ambiguity that our results of Section 3 rely on in a two-period model.

Consider an agent whose preferences over date-1 state-dependent consumption plans are described by multiple-prior expected utility with the set of probabilities $\mathcal{P}$ and linear utility function. Date-1 endowment $\tilde{e}$ is risky. There is a single asset with date-1 payoff $\tilde{x}$ and date-0 price p. At first, we assume that short-sales are unrestricted.

The investment problem is

$$\max_{c_0,h} \left[c_0 + \beta \min_{P\in\mathcal{P}} E_P(\tilde{e}+\tilde{x}h)\right], \tag{27}$$

$$\text{subject to} \quad c_0 + ph = w_0,$$

where w_0 is date-0 wealth.

For any date-1 consumption plan $\tilde{c}$ we denote the set of minimizing probabilities at $\tilde{c}$ by $\mathcal{P}(\tilde{c}) = \operatorname{argmin}_{P\in\mathcal{P}} E_P[\tilde{c}]$. A necessary and sufficient condition for h^* to be a solution to (27) is that

$$\min_{P\in\mathcal{P}(c^*)} \beta E_P[\tilde{x}] \le p \le \max_{P\in\mathcal{P}(c^*)} \beta E_P[\tilde{x}] \tag{28}$$

where $c^* = \tilde{e} + \tilde{x}h^*$. Note that (28) can be equivalently written as $p = \beta E_P[\tilde{x}]$ for some $P \in \mathcal{P}(c^*)$. The left-hand and the right-hand sides of (28) are equal if $\mathcal{P}(c^*)$ is singleton which is exactly when the multiple-prior utility is differentiable at c^*.

The proof of (28) is a simple application of superdifferential calculus. We sketch the argument for completeness. Define function $g : \Re \to \Re$ by $g(h) = \min_{P\in\mathcal{P}} E_P[\beta\tilde{e} + (\beta\tilde{x} - p)h]$. Function g is concave. A necessary and sufficient condition for h^* to be a solution to (27) is that $0 \in \partial g(h^*)$, that is, 0 lies in the superdifferential of g at h^*. It holds $\partial g(h^*) = \{\phi \in \Re : \phi = \beta E_P[\tilde{x}] - p \text{ for some } P \in \mathcal{P}(c^*)\}$. We have $0 \in \partial g(h^*)$ if and only if (28).

We focus now on the possibility of the solution being $h^* = 0$. If follows from (28) that $h^* = 0$ is a solution to (27) if and only if

$$\min_{P\in\mathcal{P}(\tilde{e})} \beta E_P[\tilde{x}] \le p \le \max_{P\in\mathcal{P}(\tilde{e})} \beta E_P[\tilde{x}] \tag{29}$$

If (29) holds with strict inequalities, then $h^* = 0$ is the unique solution.

If there is zero short-sales constraint in the investment problem (27), then the optimal investment is $h^* = 0$ if and only if $p \geq \min_{P \in \mathcal{P}(\tilde{e})} \beta E_P[\tilde{x}]$. If the inequality is strict, then $h^* = 0$ is the unique optimal investment.

These observations extend the portfolio-inertia result of Dow and Werlang (1992). If date-1 endowment $\tilde{e}$ is risk-free, then $\mathcal{P}(\tilde{e}) = \mathcal{P}$ and it follows from (29) that the optimal investment is zero for all asset prices in the interval between the minimum and the maximum discounted expected payoff over all beliefs in $\mathcal{P}$.

Figure 1 illustrates the optimal investment under ambiguity with background risk and zero short-sales constraint. There are two states and consumption takes place only at date 1. The set of probabilities is $\mathcal{P} = \{(\pi, 1-\pi) : \ 0.4 \leq \pi \leq 0.6\}$. The unique probability measure in $\mathcal{P}$ that minimizes the expected value of endowment $e_1 = (10, 5)$ is $\pi_1 = (0.4, 0.6)$. For asset price $p = E_{\pi_1}[\tilde{x}]$, holding one share of the asset - which results in consumption equal to $e_1 + (\tilde{x} - p)$ - is an optimal investments (as is zero holding). At this price p, the optimal investment for initial endowment $e_2 = (5, 10)$ and subject to zero short-sales constraint is zero. This is so because $p > E_{\pi_2}[\tilde{x}]$ where $\pi_2 = (0.6, 0.4)$ is the probability minimizing the expected value of e_2 over $\mathcal{P}$.

In a two-agent economy where both agents have the set of probabilities $\mathcal{P}$ but one has endowment e_1 while the other has e_2 and there is unitary supply of the asset traded under zero short-sales constraint, the equilibrium price of the asset is p. The first agent holds the asset. Note that $p = \max_{P \in \mathcal{P}} E_P[\tilde{x}]$.

Dynamic portfolio choice under ambiguity.

Consider the recursive multiple-prior expected utility (17). Let $\Delta u_t(c)$ denote date-t continuation utility in excess of current consumption, that is,

$$\Delta u_t(c) \equiv u_t(c) - c_t.$$

Let c^i and h^i be equilibrium consumption and asset holding of agent i. Event-s_t plan $(c^i(s_t), h^i(s_t))$ must solve

$$\max_{c(s_t), h(s_t)} \left[c(s_t) + \beta \min_{P \in \mathcal{P}(s_t)} E_P[e^i_{t+1} + (p_{t+1} + x_{t+1})h(s_t) - p_{t+1}h^i_{t+1} + \Delta u_{t+1}(c^i)|s_t] \right],$$

$$\text{subject to} \quad c(s_t) + p(s_t)h(s_t) = w^i(s_t), \quad h(s_t) \geq 0,$$

where $w^i(s_t)$ denotes agent's wealth at s_t.

Let agent's i effective belief at c^i be

$$P_t^i(s_t) = \text{argmin}_{P \in \mathcal{P}_t(s_t)} E_P[u_{t+1}(c^i)|s_t]. \tag{30}$$

A necessary condition for $h^i(s_t) > 0$ to be a solution to () is

$$\min_{P \in P_t^i(s_t)} \beta E_P[(p_{t+1} + x_{t+1})|s_t] \leq p_t(s_t) \leq \max_{P \in P_t^i(s_t)} \beta E_P[(p_{t+1} + x_{t+1})|s_t] \tag{31}$$

The respective condition for $h^i(s_t) = 0$ is

$$p(s_t) \geq \min_{P \in P_t^i(s_t)} \beta E_P[(p_{t+1} + x_{t+1})|s_t] \tag{32}$$

If there is unique effective belief $P_t^i(s_t)$, then (31) simplifies to

$$p_t(s_t) = \beta E_{P_t^i(s_t)}[(p_{t+1} + x_{t+1})|s_t], \tag{33}$$

If $\mathcal{P}_t(s_t)$ is a polyhedral set as in eq. (16), there is a unique minimizing probability in (30) for almost every continuation utility.

Proof of Theorem 1: Let the set of one-period-ahead probabilities $\mathcal{P}_t(s_t)$ be given by eq. (16). The set of effective beliefs $P_t^1(s_t)$ can be characterized as follows. First, it is easy to see that the probability that minimizes the expected value of next-period endowment e_{t+1}^1 is $Q^1(s_t)$. Continuation utility $u_{t+1}(c^1)$ differs from endowment by the term $(p_{t+1} + x_{t+1})h^1(s_t) - p_{t+1}h_{t+1}^1 + \Delta u_{t+1}(c^1)$. The first part $(p_{t+1} + x_{t+1})h^1(s_t) - p_{t+1}h_{t+1}^1$ is the net-payoff of portfolio strategy. It depends on dividends, but not on the agent's endowment. The same holds for excess continuation utility $\Delta u_{t+1}(c^1)$. It depends on dividends, but not on the agent's endowment. Therefore the minimizing probabilities for next-period continuation utility are the same as those for next-period endowment. Therefore the effective belief of agent 1 is $Q^1(s_t)$. Similarly, the effective belief of agent 2 is $Q^2(s_t)$. This concludes step 1.

For step 2, consider asset prices defined by the recursive relation (24), that is

$$p_t(s_t) = \beta\big[(1 - \hat{q}_t(s_t))p_{t+1}(s_t, 0) + \hat{q}_t(s_t)(p_{t+1}(s_t, 1) + 1)\big], \tag{34}$$

Prices $p(s_t)$ depend only on on the number of high dividends from date 1 through t. We shall prove that

$$p_{t+1}(s_t,1)+1>p_{t+1}(s_t,0) \tag{35}$$

Inequality (35) implies that $Q_t^i(s_t)$ is not the maximizing probability in (19) for i such that $q_t^i(s_t)<\hat{q}_t(s_t)$, and hence that p is an equilibrium price and $\hat{P}_t(s_t)\neq Q_t^i(s_t)$.

In order to prove (35) we follow Harrison and Kreps (1978) and define $p_t^n(s_t)$ inductively by

$$p_t^{n+1}(s_t)=\beta\big[(1-\hat{q}_t(s_t))p_{t+1}^n(s_t,0)+\hat{q}_t(s_t)(p_{t+1}^n(s_t,1)+1)\big], \tag{36}$$

with $p_t^0(s_t)=0$. Using an inductive argument, one can show that $p_t^n(s_t)$ is bounded by $\beta/(1-\beta)$ for every n and every s_t, and non-decreasing in n. Therefore $\lim_n p_t^n(s_t)=p_t(s_t)$. We claim that (35) holds for p^n for every n. The proof is by induction. Assume that (35) holds for p^n. We obtain

$$p_{t+1}^{n+1}(s_t,1)+1-p_{t+1}^{n+1}(s_t,0)\geq 1+\beta p_{t+2}^n(s_t,1,0)-\beta(1+p_{t+2}^n(s_t,0,1), \tag{37}$$

where we used (36). Since $p_{t+2}^n(s_t,0,1)=p_{t+2}^n(s_t,1,0)$, we obtain $p_{t+1}^{n+1}(s_t,1)+1-p_{t+1}^{n+1}(s_t,0)\geq 1-\beta>0$. Taking limits, we obtain (35).

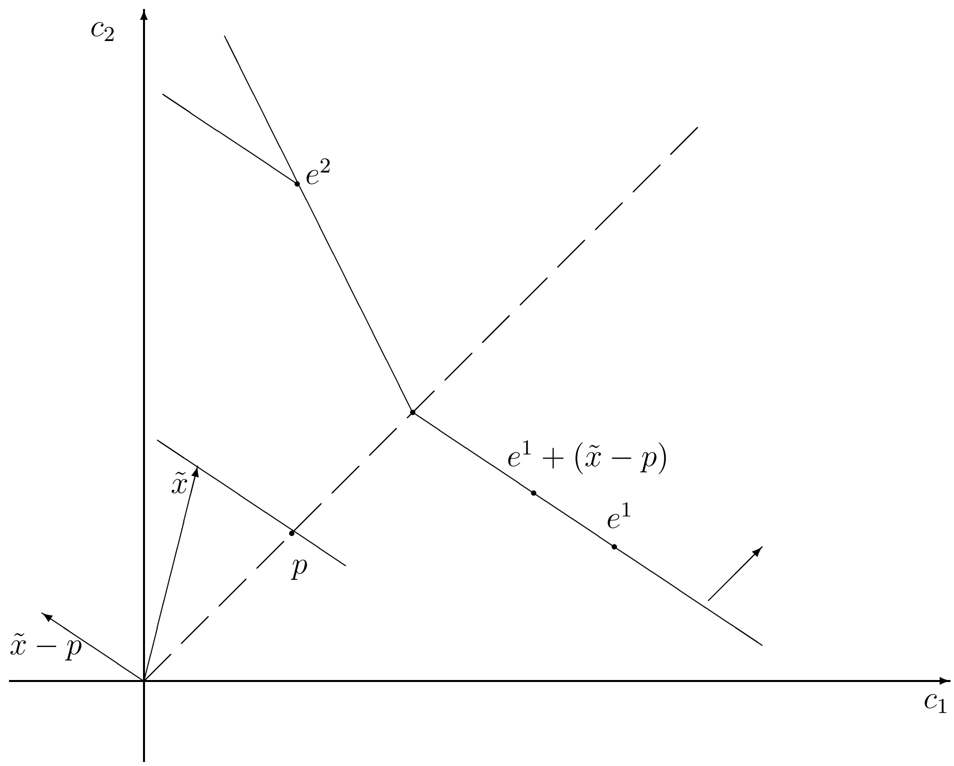

Figure 1: Equilibrium under ambiguity

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