# Optimal Portfolio Choice under Decision-Based Model Combinations

Davide Pettenuzzo
Brandeis University

Francesco Ravazzolo
Norges Bank
BI Norwegian Business School

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# Stock market predictability

- Economic value of return predictability is questionable, see Bossaerts and Hillion (199) and Welch and Goyal (2008).
- Recent empirical evidence, see Avramov (2002), Aiolfi and Favero (2005), Rapach et al. (2010) and Dangl and Halling (2012), shows an important role for model uncertainty and model combinations improve out-of-sample predictability.
- In particular, Avramov (2002) and Dangl and Halling (2012) propose Bayesian Model Averaging:

$$p(r_{t+1}|\mathcal{D}^t) = \sum_{i=1}^{N} P\left( M_i | \mathcal{D}^t \right) p(r_{t+1}|M_i, \mathcal{D}^t) \quad (1)$$

where $p(r_{t+1}|M_i, \mathcal{D}^t)$ is the predictive density for $r_{t+1}$ from model i, $P\left( M_i | \mathcal{D}^t \right)$ is the posterior probability of model $i$, derived by Bayes' rule,

$$P\left( M_i | \mathcal{D}^t \right) = \frac{P\left( \mathcal{D}^t | M_i \right) P\left( M_i \right)}{\sum_{j=1}^{N} P\left( \mathcal{D}^t | M_j \right) P\left( M_j \right)}, \qquad i = 1, ..., N \quad (2)$$

# Motivation: combination issues

- Averaging as tool to improve forecast accuracy (Barnes (1963), Bates and Granger (1969)).
- Parameter and model uncertainties play an important role (BMA, Roberts (1965)).
- Model performance varies over time with some persistence (Diebold and Pauly (1987), Guidolin and Timmermann (2009), Hoogerheide et al. (2010), Gneiting and Raftery (2007); Del Negro, Hasegawa and Schorfheide (2013)).
- Model performances might differ over regions of interest/quantiles (mixture of predictives; generalized LOP: Fawcett, Kapetanios, Mitchell and Price, 2014).
- Model set is possible incomplete (Geweke (2009), Geweke and Amisano (2010), Waggoner and Zha (2011)).
- Optimal estimated weights incomplete (Hall and Mitchell (2007), Geweke and Amisano (2010), Conflitti, De Mol, Giannone (2012)).
- Individual models should be weighted based on how they fare relative to the final objective function of the investor, see also Herman's presentation for nowcasting.

# Contribution

In the spirit of Pesaran and Skouras (2007) and evidence in Cenesizoglu and Timmermann (2012), we propose a Decision-Based Density Combination approach (DB-DeCo) that:

- Combines the entire predictive densities of the individual models;
- Allows for model incompleteness;
- Estimate optimal time-varying weights given the final objective function, that is a utility-based objective function summarizing investment portfolio past performance.

# Contribution

We apply the DB-DeCo to predict and invest on monthly S&P500 stock index returns over the sample 1947-2010 and combine:

1. A set of linear predictive regressions for stock returns, each including as regressor one of the predictor variables used by Goyal and Welch (2008).
2. A set of predictive densities from time-varying parameters and stochastic volatility (TVP-SV) models (extending Johannes et al. (2013), Dang and Halling (2012)).

We find that:

- DB-DeCo leads to substantial improvements in the predictive accuracy of the equity premium relative to individual models and other combination schemes.
- In the benchmark case of a power utility investor with relative risk aversion of five, DB-DeCo method yields an annualized Certainty Equivalent Return (CER) of 94 basis points higher than the prevailing mean (PM) model, while BMA delivers a negative annualized CER and equal weight combination just 2 basis points higher.
- Allowing for TVP-SV in the DB-DeCo method results in an increase in CER of more than 150 basis points.

# Decision-Based Density Combination: Problem

The combination problem can be written as:

$$p(r_{t+1}|\mathcal{D}^t) = \int p(r_{t+1}|\widetilde{\mathbf{r}}_{t+1}, \mathbf{w}_{t+1}, \mathcal{D}^t) p(\mathbf{w}_{t+1}|\widetilde{\mathbf{r}}_{t+1}, \mathcal{D}^t) p\left(\widetilde{\mathbf{r}}_{t+1}\middle| \mathcal{D}^t\right) d\widetilde{\mathbf{r}}_{t+1} d\mathbf{w}_{t+1}$$

- Incomplete set of models in $p(r_{t+1}|\widetilde{\mathbf{r}}_{t+1}, \mathbf{w}_{t+1}, \mathcal{D}^t)$ by specifying a stochastic relationship between individual densities and combined densities.
- Time-varying weights weights in $p(\mathbf{w}_{t+1}|\widetilde{\mathbf{r}}_{t+1}, \mathcal{D}^t)$.
- Learning mechanism in $p(\mathbf{w}_{t+1}|\widetilde{\mathbf{r}}_{t+1}, \mathcal{D}^t)$ based on a utility-based objective function.

We follow Billio et al. (2013) and apply a Gaussian combination, with Logistic-Gaussian Weights and extend them with a learning mechanism based on the Certainty Equivalent Return (CER).

# Decision-Based Density Combination: Scheme

Conditional combination scheme:

$$p(r_{t+1}|\widetilde{\mathbf{r}}_{t+1}, \mathbf{w}_{t+1}, \sigma_{\kappa}^{-2}) \propto \exp\left\{-\frac{1}{2}\left(r_{t+1} - \widetilde{\mathbf{r}}_{t+1}\mathbf{w}_{t+1}\right)' \sigma_{\kappa}^{-2}\left(r_{t+1} - \widetilde{\mathbf{r}}_{t+1}\mathbf{w}_{t+1}\right)\right\}$$

where the weights are logistic transforms

$$\mathbf{w}_{t+1} = (g_1(z_{1,t+1}), \ldots, g_N(z_{N,t+1}))'$$

The latent processes $\mathbf{z}_{t+1}$ evolve over time and map into the combination weights $\mathbf{w}_{t+1}$ as:

$$\begin{aligned}\mathbf{z}_{t+1} &\sim p(\mathbf{z}_{t+1}|\mathbf{z}_t, \Delta\zeta_t, \Lambda) \\ &\propto |\Lambda|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\left(\mathbf{z}_{t+1} - \mathbf{z}_t - \Delta\zeta_t\right)' \Lambda^{-1}\left(\mathbf{z}_{t+1} - \mathbf{z}_t - \Delta\zeta_t\right)\right\}\end{aligned}$$

where $\zeta_t$ depends on the final objective function of the investor up to time t:

$$\zeta_{i,t} = (1-\lambda)\sum_{\tau=\underline{t}}^{t} \lambda^{t-\tau} f\left(r_\tau, \widetilde{\mathbf{r}}_{i,\tau}\right), \quad i = 1, \ldots, N$$

# Decision-Based Density Combination: CER based weights

A power utility investor who at time $\tau - 1$ chooses a portfolio by allocating her wealth $W_{\tau-1}$ between the riskless asset and one risky asset, and subsequently holds onto that investment for one period, her CER is given by

$$f\left(r_\tau, \widetilde{\mathbf{r}}_{i,\tau}\right) = \left[(1-A)\, U\left(W^*_{i,\tau}\right)\right]^{1/(1-A)}$$

where

$$U\left(W^*_{i,\tau}\right) = \frac{\left[\left(1-\omega^*_{i,\tau-1}\right)\exp\left(r^f_{\tau-1}\right) + \omega^*_{i,\tau-1}\exp\left(r^f_{\tau-1} + r_\tau\right)\right]^{1-A}}{1-A}$$

$r^f_{\tau-1}$ denotes the continuously compounded Treasury bill rate at time $\tau - 1$, $A$ stands for the investor's relative risk aversion, $r_\tau$ is the realized excess return at time $\tau$, and $\omega^*_{i,\tau-1}$ denotes the optimal allocation to stocks according to the prediction made for $r_\tau$ by model $M_i$,

$$\omega^*_{i,\tau-1} = \arg\max_{\omega_{\tau-1}} \int U\left(\omega_{\tau-1}, r_\tau\right) p(r_\tau | M_i, \mathcal{D}^{\tau-1}) dr_\tau \qquad (3)$$

# Decision-Based Density Combination: Individual predictive densities

Bayesian Linear Regression Model (Welch and Goyal (2008))

$$
\begin{aligned}
r_{\tau+1} &= \mu + \beta x_{\tau} + \varepsilon_{\tau+1}, \quad \tau = 1, ..., t-1, \\
\varepsilon_{\tau+1} &\sim N(0, \sigma_{\varepsilon}^2).
\end{aligned}
$$

Bayesian Time-Varying Parameter Stochastic Volatility Model (Johannes et al. (2013))

$$
r_{\tau+1} = (\mu + \mu_{\tau+1}) + (\beta + \beta_{\tau+1})\, x_{\tau} + \exp(h_{\tau+1})\, u_{\tau+1}, \quad \tau = 1, ..., t-1,
$$

$$
\begin{bmatrix} \mu_{\tau+1} \\ \beta_{\tau+1} \end{bmatrix} = \begin{bmatrix} \gamma_{\mu} & 0 \\ 0 & \gamma_{\beta} \end{bmatrix} \begin{bmatrix} \mu_{\tau} \\ \beta_{\tau} \end{bmatrix} + \begin{bmatrix} \eta_{1,\tau+1} \\ \eta_{2,\tau+1} \end{bmatrix},
$$

$$
h_{\tau+1} = \lambda_0 + \lambda_1 h_{\tau} + \xi_{\tau+1}
$$

# Decision-Based Density Combination: Estimation

- Non linear state space model with observation equation the combination equation, and with nonlinear latent equation the weights.
- Apply a Sequential Monte Carlo algorithm to estimate it;
- using a modification of the GPU toolbox in Casarin et al. (2013).

# Application

- We focus on the S&P500 stock index return in excess of short T-bill rate.
- 15 regressors as in Welch and Goyal (2008).
- 5 combination chemes: EW, BMA, Optimal Pooling, DeCO, DB-DeCo.
- We initially estimate our regression models over the period January 1927–December 1946, and use the estimated coefficients to forecast excess returns for January 1947.
- OOS sample: January 1947-December 2010.
- Statical accuracy evaluation: out-of-sample $R^2$; cumulative rank probability score differentials and log predictive score differentials. DM test with a serial correlation-robust variance.
- Economic performance: CER for portfolio investment decisions based on recursive out-of-sample forecasts of monthly excess returns.
- Results in the presentation are for the case with $A = 5$ and $\tau = 0.95$.

# Out-of-sample point ($R^2$) forecast performance

| | Linear | | TVP-SV | |
|---|---|---|---|---|
| Log dividend yield | -0.44% | | **0.99%** | * |
| Log earning price ratio | -2.27% | | -0.07% | |
| Log smooth earning price ratio | -1.51% | | **0.68%** | |
| Log dividend-payout ratio | -1.91% | | -1.84% | |
| Book-to-market ratio | -1.79% | | -0.20% | |
| T-Bill rate | -0.12% | | **0.18%** | |
| Long-term yield | -0.95% | | -1.05% | |
| Long-term return | -1.55% | | -0.70% | |
| Term spread | **0.09%** | | **0.04%** | |
| Default yield spread | -0.24% | | -0.22% | |
| Default return spread | -0.23% | | -0.47% | |
| Stock variance | **0.09%** | | -0.99% | |
| Net equity expansion | -0.93 % | | -0.88% | |
| Inflation | -0.19% | | -0.20% | |
| Log total net payout yield | -0.79% | | **0.09%** | |
| Combinations | | | | |
| Equal weighted combination | **0.49%** | | **0.62%** | ** |
| BMA | **0.39%** | | **0.41%** | |
| Optimal prediction pool | -1.93% | | -0.86% | |
| Density combination | **0.43%** | | **1.33%** | *** |
| Decision-based density combination | **2.32%** | *** | **2.13%** | *** |

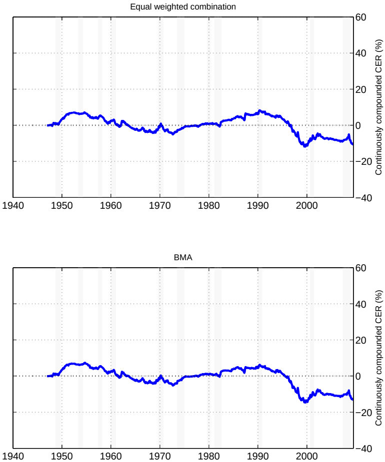
# Out-of-sample density forecast performance

| | CRPSD | | | | LSD | | | |
|---|---|---|---|---|---|---|---|---|
| | Linear | | TVP-SV | | Linear | | TVP-SV | |
| Log dividend yield | -0.37% | | **8.21%** | *** | -0.15% | | **11.18%** | *** |
| Log earning price ratio | -0.79% | | **8.05%** | *** | -0.17% | | **11.24%** | *** |
| Log smooth earning price ratio | -0.59% | | **8.40%** | *** | **0.02%** | | **11.49%** | *** |
| Log dividend-payout ratio | -0.45% | | **6.64%** | *** | -0.19% | | **9.38%** | *** |
| Book-to-market ratio | -0.61% | | **8.25%** | *** | -0.12% | | **11.38%** | *** |
| T-Bill rate | -0.07% | | **7.17%** | *** | -0.10% | | **9.17%** | *** |
| Long-term yield | -0.38% | | **6.91%** | *** | -0.22% | | **9.48%** | *** |
| Long-term return | -0.46% | | **6.70%** | *** | -0.14% | | **9.06%** | *** |
| Term spread | **0.08%** | | **6.98%** | *** | -0.03% | | **8.87%** | *** |
| Default yield spread | -0.07% | | **7.17%** | *** | -0.08% | | **9.43%** | *** |
| Default return spread | -0.11% | | **7.03%** | *** | -0.03% | | **9.37%** | *** |
| Stock variance | **0.02%** | | **8.38%** | *** | -0.02% | | **11.86%** | *** |
| Net equity expansion | 0.00% | | **7.22%** | *** | **0.04%** | | **9.36%** | *** |
| Inflation | -0.05% | | **7.56%** | *** | -0.15% | | **10.01%** | *** |
| Log total net payout yield | -0.33% | | **7.16%** | *** | **0.06%** | | **9.74%** | *** |
| Combinations | | | | | | | | |
| Equal weighted combination | **0.08%** | | **7.88%** | *** | -0.11% | | **10.49%** | *** |
| BMA | **0.10%** | | **6.22%** | *** | **0.03%** | | **10.40%** | *** |
| Optimal prediction pool | -0.43% | | **8.36%** | *** | -0.11% | | **11.81%** | *** |
| Density combination | **0.07%** | | **8.53%** | *** | **0.00%** | | **11.17%** | *** |
| Decision-based density combination | **0.73%** | *** | **9.26%** | *** | **0.26%** | *** | **11.75%** | *** |

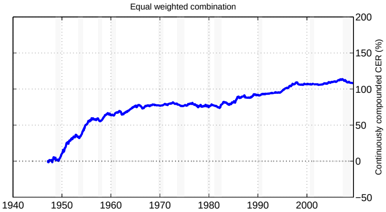
# Certainty Equivalent Returns

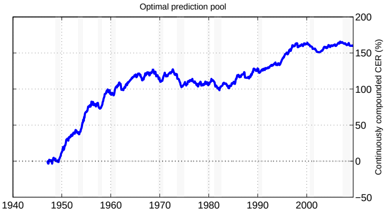
| | Linear | TVP-SV |
|---|---|---|
| Log dividend yield | -0.33% | **0.90%** |
| Log earning price ratio | **0.25%** | **0.91%** |
| Log smooth earning price ratio | -0.38% | **0.92%** |
| Log dividend-payout ratio | **0.41%** | **0.96%** |
| Book-to-market ratio | -0.58% | **0.71%** |
| T-Bill rate | -0.26% | **0.79%** |
| Long-term yield | -0.34% | **0.50%** |
| Long-term return | -0.42% | **0.77%** |
| Term spread | **0.15%** | **0.89%** |
| Default yield spread | -0.20% | **0.90%** |
| Default return spread | -0.14% | **0.64%** |
| Stock variance | **0.00%** | **0.98%** |
| Net equity expansion | -0.14% | **0.76%** |
| Inflation | -0.17% | **0.76%** |
| Log total net payout yield | -0.37% | **0.47%** |
| Combinations | | |
| Equal weighted combination | **0.02%** | **1.17%** |
| BMA | -0.05% | **1.03%** |
| Optimal prediction pool | -0.82% | **0.96%** |
| Density combination | -0.01% | **1.74%** |
| Decision-based density combination | **0.94%** | **2.49%** |

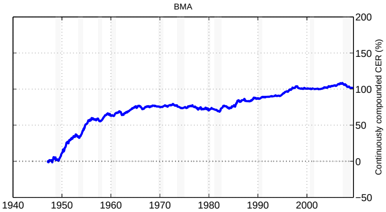
# Cumulative Certainty Equivalent Returns, Linear Models

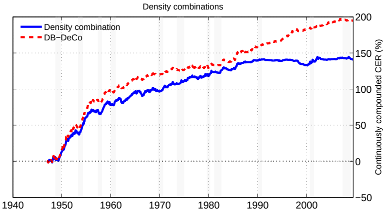


# Cumulative Certainty Equivalent Returns, TVP-SV



# Robustness

- Different risk aversion coefficients, $A = 2$ and $A = 10$.
- Subsamples: NBER expansions, NBER recessions, 1947-1978 and 1979-2010.
- Alternative learning dynamics, $\tau = 0.9$.
- Mean Variance preferences.
- Alternative individual model priors: dispersed prior distributions and more concentrated prior distributions.
- Alternative DB-DeCo priors: degree of time variation in the DB-DeCo combination weights.

# Certainty Equivalent Returns, alternative RW-SV benchmark

| | A=2 | | A=5 | | A=10 | |
|---|---|---|---|---|---|---|
| | Linear | TVP-SV | Linear | TVP-SV | Linear | TVP-SV |
| Log dividend yield | -2.14% | -0.04% | -1.23% | **0.00%** | -0.62% | -0.00% |
| Log earning price ratio | -1.04% | **0.36%** | -0.65% | **0.02%** | -0.33% | **0.05%** |
| Log smooth earning price ratio | -2.52% | **0.06%** | -1.28% | **0.02%** | -0.65% | **0.02%** |
| Log dividend-payout ratio | -0.16% | -0.02% | -0.49% | **0.06%** | -0.25% | -0.03% |
| Book-to-market ratio | -2.53% | **0.25%** | -1.47% | -0.19% | -0.74% | -0.12% |
| T-Bill rate | -1.82% | **0.08%** | -1.15% | -0.10% | -0.59% | -0.05% |
| Long-term yield | -2.02% | -0.33% | -1.24% | -0.39% | -0.63% | -0.19% |
| Long-term return | -1.97% | -0.21% | -1.31% | -0.13% | -0.65% | -0.06% |
| Term spread | -0.68% | **0.48%** | -0.75% | -0.01% | -0.40% | -0.01% |
| Default yield spread | -1.64% | -0.07% | -1.10% | -0.00% | -0.56% | -0.00% |
| Default return spread | -1.22% | -0.09% | -1.04% | -0.26% | -0.55% | -0.12% |
| Stock variance | -1.14% | **0.16%** | -0.90% | **0.09%** | -0.44% | **0.06%** |
| Net equity expansion | -0.62% | **0.04%** | -1.03% | -0.14% | -0.54% | -0.07% |
| Inflation | -1.56% | -0.25% | -1.07% | -0.13% | -0.53% | -0.06% |
| Log total net payout yield | -2.23% | -0.76% | -1.26% | -0.43% | -0.64% | -0.22% |
| Equal weighted combination | -1.10% | **0.18%** | -0.88% | **0.27%** | -0.44% | **0.13%** |
| BMA | -1.13% | **0.05%** | -0.59% | **0.13%** | -0.28% | **0.05%** |
| Optimal prediction pool | -2.18% | **0.15%** | -1.72% | **0.06%** | -0.87% | **0.02%** |
| Density combination | -1.16% | **0.57%** | -0.91% | **0.84%** | -0.45% | **0.42%** |
| Decision-based density combination | **1.47%** | **1.20%** | **0.04%** | **1.59%** | **0.04%** | **0.79%** |

# Conclusion

- Decision-Based Density Combination approach for stock market predictability and asset allocation:
  - Combines the entire predictive densities of the individual models;
  - Allows for model incompleteness;
  - Estimate optimal time-varying weights given the final objective function, that is a utility-based objective function.
- Results show that:
  - DB-DeCo leads to substantial improvements in the predictive accuracy of the equity premium.
  - Large CER gains, even against the PM with stochastic volatility.