Macroeconomic Dynamics Near the ZLB: A Tale of Two Equilibria *

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Abstract

This paper studies the dynamics of a New Keynesian DSGE model near the zero lower bound (ZLB) on nominal interest rates. In addition to the standard targeted-inflation equilibrium, we consider a deflation equilibrium as well as a Markov sunspot equilibrium that switches between a targeted-inflation and a deflation regime. We use the particle filter to estimate the state of the U.S. economy during and after the 2008-09 recession under the assumptions that the U.S. economy has been in either the targeted-inflation or the sunspot equilibrium. We consider a combination of fiscal policy (calibrated to the American Recovery and Reinvestment Act) and monetary policy (that tries to keep interest rates near zero) and compute government spending multipliers. Ex-ante multipliers (cumulative over one year) under the targeted-inflation regime are around 0.9. A monetary policy that keeps interest rates at zero can raise the multiplier to 1.7. The ex-post (conditioning on the realized shocks in 2009-2011) multiplier is estimated to be 1.3. Conditional on the sunspot equilibrium the multipliers are generally smaller and the scope for conventional expansionary monetary policy is severely limited. JEL CLASSIFICATION: C5, E4, E5

KEY WORDS: DSGE Models, Government Spending Multiplier, Multiple Equilibria, Nonlinear Filtering, Nonlinear Solution Methods, ZLB

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1 Introduction

Investors' access to money, which is an asset that in addition to providing transaction services yields a zero nominal return, prevents nominal interest rates from falling below zero and thereby creates a zero lower bound (ZLB) for nominal interest rates. Traditionally, the ZLB has mostly been ignored in the specification of dynamic stochastic general equilibrium (DSGE) models that are tailored toward the analysis of U.S. monetary and fiscal policy. However, since the beginning of 2009 the U.S. Federal Funds rate has been effectively zero and the literature on the analysis of DSGE models with an explicit ZLB constraint has been growing rapidly. Our paper contributes to this literature by solving for multiple equilibria in a New Keynesian DSGE model with a full set of stochastic shocks using global projection methods; by fitting the model to U.S. data on output growth, inflation, and interest rate and using a filter to estimate the shocks that generated the 2008-09 recession and provided the initial conditions for the subsequent recovery; and by assessing the effectiveness of expansionary fiscal and monetary policy during the Great Recession.

Once the ZLB is explicitly included in a monetary model with an interest-rate feedback rule, there typically exist multiple equilibria. The most-widely studied equilibrium is the one in which the economy fluctuates around the steady state where actual inflation coincides with the central bank's inflation target (targeted-inflation equilibrium). Our paper is the first to study two additional equilibria in a nonlinear DSGE model with a full set of structural shocks: a minimal-state-variable equilibrium in which the endogenous variables fluctuate near a steady state with zero interest rates (deflation equilibrium); and an equilibrium in which the economy alternates between a targeted-inflation regime and a deflation regime according to the realization of a non-fundamental Markov-switching process (sunspot equilibrium). For the empirical analysis, we focus on the targeted-inflation equilibrium and the sunspot equilibrium because the U.S. never experienced a prolonged period of deflation as the deflation equilibrium would predict.

The multiplicity of equilibria and a potential switch to a deflation regime that resembles the economic experience of Japan since the late 1990's was a real concern to U.S. policy makers in the beginning of 2009, e.g. Bullard (2010). Thus, since the Great Recession the question of how economic dynamics and the effects of fiscal and monetary policy differ in the deflation regime has been very important. Our quantitative analysis provides an ex-ante assessment conditional on the state of the U.S. economy at the end of 2009:Q1. With hindsight, our empirical analysis reveals that the probability of a switch to a deflation regime is low. This conclusion crucially relies on our ability to solve for the sunspot equilibrium.

Using data from 1984 to 2007, a period in which the ZLB was non-binding, we estimate the parameters of our DSGE model, based on a second-order perturbation solution, which turns out be essentially identical to the projection solution of the model for values of the state variables that were empirically relevant prior to the 2008-09 recession. Conditioning on the parameter estimates we then switch to the projection solution and use a nonlinear filter to extract the exogenous shock processes, that provide the initial conditions for our policy experiments. We do so for the targeted-inflation equilibrium and the sunspot equilibrium. Under the sunspot equilibrium there is a non-zero probability of a switch to the deflation regime in the beginning of 2009. We study the effects of an increase in government spending that is calibrated to match the size of the federal contracts, grants, and loans portion of the American Recovery and Reinvestment Act (ARRA), which was signed into law in 2009:Q1. We also consider a concurrent expansionary monetary policy that keeps interest rates near zero for an extended period of time.

We consider two types of policy exercises, which we label as ex ante and ex post. In the ex-ante analysis we take the states at the end of 2009:Q1 as given and simulate the model economy forward, with and without the policy intervention. We find that the ex-ante cumulative fiscal multiplier both under the targeted-inflation and the sunspot equilibrium is about 0.9 over a one-year horizon. Under the targeted-inflation equilibrium the nominal interest rises quickly and the economy moves away from the ZLB. This creates scope for expansionary monetary policy that can amplify the effect of the fiscal expansion and raise the multiplier from 0.9 to 1.7. For the analysis under the sunspot equilibrium we assume that the economy is initially in the deflation state, which implies, looking forward, that interest rates are likely to stay close to zero and the scope for conventional monetary stimulus is much smaller than in the targeted-inflation equilibrium.

In the ex-post analysis we condition on the actual filtered shocks from the years 2009 and 2010. Counterfactual outcomes are computed by removing the contribution of expansionary fiscal and monetary policy actions from the filtered shocks. In the absence of the economic stimulus, output growth in 2009:Q2 would have been substantially lower and the economy would have experienced a prolonged deflationary episode through the end of 2010. The ex-post policy multiplier for the expansionary fiscal policy is larger than ex ante, around 1.3 over a one-year horizon. During 2009 and 2010 the realized shocks pushed the economy closer to the ZLB. Once at the ZLB, the feedback portion of the monetary policy rule is inactive and the fiscal stimulus is not accompanied by a rise in nominal interest rates. The resulting lower real rates stimulate demand and amplify the fiscal policy effect. This mechanism is emphasized, for instance, by Eggertsson and Woodford (2003) and Christiano, Eichenbaum, and Rebelo (2011). However, according to our empirical analysis it is not as strong as these authors claim.

The ex-post multiplier on a combined fiscal and monetary intervention is identical to the ex-ante multiplier over a one-quarter horizon, but rises from 1.3 to 1.6 over four quarters and from 1.2 to 2.0 over eight quarters. In other words, at the beginning of 2009, in the logic of the targeted-inflation equilibrium, the U.S. central bank had no leverage to stimulate the economy with conventional monetary policy. By the second half of 2010 the actual monetary policy was expansionary in the sense that the model-implied feedback rule would have predicted a positive interest rate. This expansionary monetary policy amplified the effect of the fiscal stimulus. The multipliers for the sunspot equilibrium are quite similar, because the data suggests that the economy quickly reverted back to the targeted-inflation regime.

Our paper is related to several strands of the literature. It has been well-known that monetary DSGE models with an explicit ZLB constraint deliver multiple equilibria. This issue has been discussed, for instance, by Benhabib, Schmitt-Grohé, and Uribe (2001a,b) and more recently in Schmitt-Grohe and Uribe (2012). In a nutshell, the relationship between nominal interest rates and inflation in a DSGE model are characterized by a consumption Euler equation and a monetary policy rule. The kink in the monetary policy rule induced by the ZLB tends to generate two pairs of steady-state interest and inflation rates that solve both equations. One can construct equilibria in the neighborhood of the two steady states as well as sunspot equilibria with a Markov-switching shock that moves the economy from the vicinity of one steady state to the vicinity of the other steady state. Such a sunspot equilibrium has been recently analyzed by Mertens and Ravn (2013), but in a model with a much more restrictive exogenous shock structure. Our paper is the first to compute a sunspot equilibrium in a New Keynesian DSGE model that is rich enough to track U.S. macroeconomic time series.

In terms of solution method, our work is most closely related to the papers by Judd, Maliar, and Maliar (2010), Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2012), and Gust, Lopez-Salido, and Smith (2012).[1] All three of these papers use global projection methods to approximate agents' decision rules in a New Keynesian DSGE model with a ZLB constraint. However, these papers solely consider the targeted-inflation equilibrium and some details of the implementation of the solution algorithm are different. To improve the accuracy of the model solution under our three equilibria, we introduce two novel features. First, we use a piece-wise smooth approximation with two separate functions characterizing the decisions when the ZLB is binding and when it is not. The above-referenced papers, on other hand, use smooth approximations with a single function covering the whole state space. This means all our decision rules allow for kinks at points in the state space where the ZLB becomes binding. Second, when constructing a grid of points in the models' state space for which the equilibrium conditions are explicitly evaluated by the projection approach, we combine draws from the ergodic distribution of the DSGE model with values of the state variables obtained by applying our filtering procedure.

[1]Most of the other papers that study DSGE models with a ZLB constraint take various shortcuts to solve the model. In particular, following Eggertsson and Woodford (2003), many authors assume that an exogenous Markov-switching process pushes the economy to the ZLB. The subsequent exit from the ZLB is exogenous and occurs with a pre-specified probability. The absence of other shocks makes it impossible to use the model to track actual data. Unfortunately, model properties tend to be very sensitive to the approximation technique and to implicit or explicit assumptions about the probability of leaving the ZLB, see Braun, Körber, and Waki (2012) and Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2012).

Our modification of the ergodic-set method proposed by Judd, Maliar, and Maliar (2010) ensures that the model solution is accurate in a region of the state space that is unlikely ex ante under the ergodic distribution of the model, but very important ex post to explain the observed data. Gust, Lopez-Salido, and Smith (2012) also use a nonlinear filter to extract shocks that allow their DSGE model to track U.S. data throughout the Great Recession. However, their empirical analysis focuses on the extent to which the ZLB constrained the ability of monetary policy to stabilize the economy.

The effect of an increase in government spending when the economy is at the ZLB is studied by Braun, Körber, and Waki (2012), Christiano, Eichenbaum, and Rebelo (2011), Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2012), Eggertsson (2009), and Mertens and Ravn (2013). Christiano, Eichenbaum, and Rebelo (2011) argue that the fiscal multiplier at the ZLB can be substantially larger than one. In general, the government spending multiplier crucially depends on whether the expansionary fiscal policy triggers an exit from the ZLB. The longer the exit from the ZLB is delayed, the larger the government spending multiplier. We capture this effect through the combination of expansionary fiscal and monetary policies and in our ex-post analysis. Mertens and Ravn (2013) emphasize that in what we would call a deflation equilibrium, the effects of expansionary government spending can be substantially smaller from the effects in the standard targeted-inflation equilibrium.

The remainder of the paper is organized as follows. Section 2 presents a simple two-equation model that we use to illustrate the multiplicity of equilibria in monetary models with ZLB constraints. We also highlight the types of equilibria studied in this paper. The New Keynesian model that is used for the quantitative analysis is presented in Section 3 and the solution of the model is discussed in Section 4. Section 5 contains the quantitative analysis and Section 6 concludes. Detailed derivations, descriptions of algorithms, and additional quantitative results are summarized in an Online Appendix.

2 A Two-Equation Example

We begin with a simple two-equation example to illustrate the types of equilibria that arise if a ZLB constraint is imposed in a monetary DSGE model with an interest-rate feedback rule. The example is adapted from Benhabib, Schmitt-Grohé, and Uribe (2001a) and Hursey and Wolman (2010). Suppose that the economy can be described by the Fisher relationship

$$R_t = r\mathbb{E}_t[\pi_{t+1}] \tag{1}$$

and the monetary policy rule

$$R_t = \max\left\{1,\ r\pi_*\left(\frac{\pi_t}{\pi_*}\right)^{\psi}\exp[\sigma\epsilon_t]\right\},\quad \epsilon_t \sim iidN(0,1),\quad \psi > 1. \tag{2}$$

Here R_t denotes the gross nominal interest rate, π_t is the gross inflation rate, and ϵ_t is a monetary policy shock. The gross nominal interest rate is bounded from below by one. Throughout this paper we refer to this bound as ZLB because it bounds the net interest rate from below by zero. Combining (1) and (2) yields a nonlinear expectational difference equation for inflation

$$\mathbb{E}_t[\pi_{t+1}] = \max\left\{\frac{1}{r},\ \pi_*\left(\frac{\pi_t}{\pi_*}\right)^{\psi}\exp[\sigma\epsilon_t]\right\}. \tag{3}$$

This model has two steady states ($\sigma = 0$), which we call the targeted-inflation steady state and the deflation steady state, respectively. In the targeted-inflation steady state inflation equals π_* and the nominal interest rate is $R = r\pi_*$. In the deflation steady state inflation equals $\pi_D = 1/r$ and the nominal interest is $R_D = 1$.

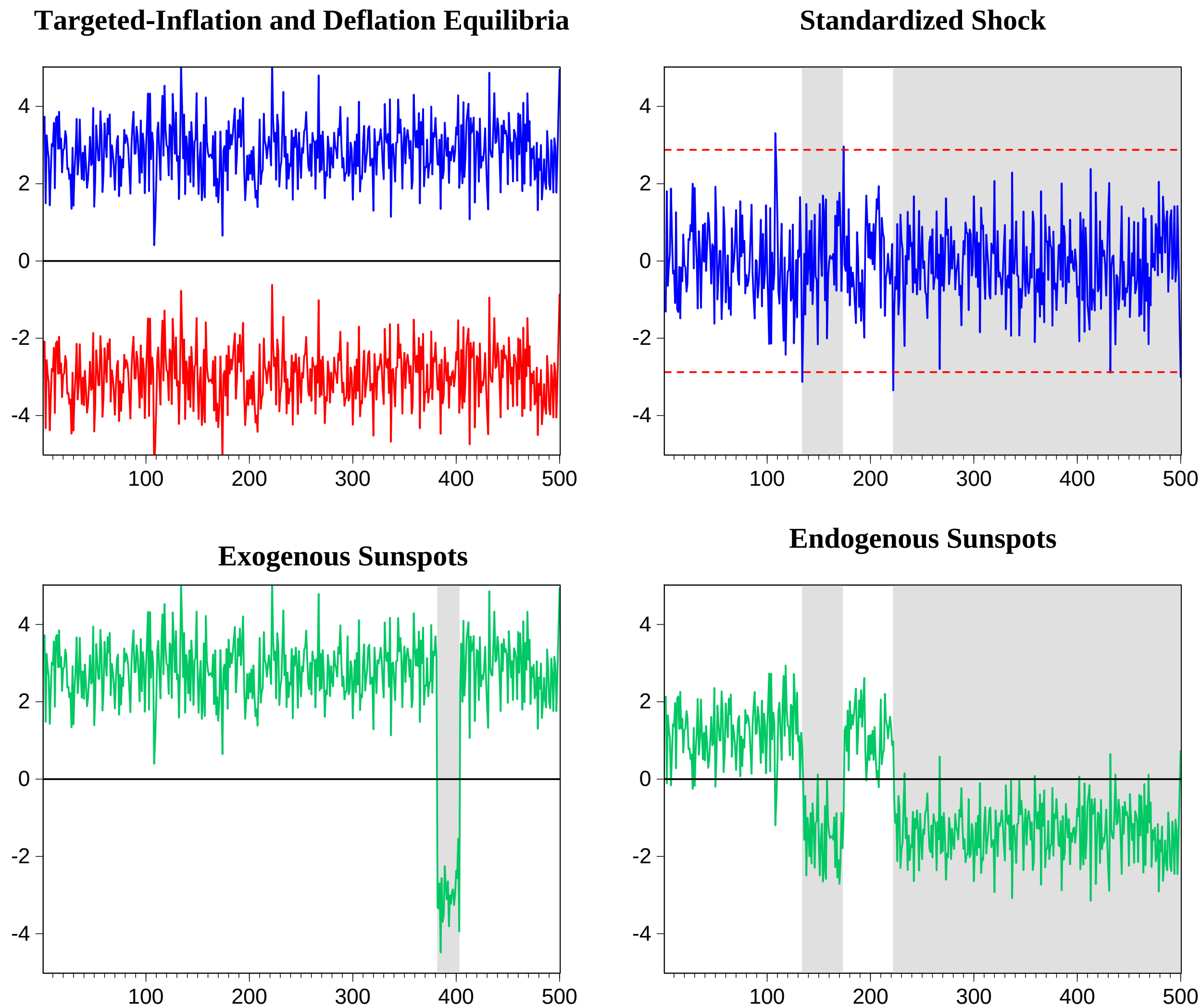
The presence of two steady states suggests that the nonlinear rational expectation difference equation (3) has multiple stable stochastic solutions. We find solutions to this equation using a guess-and-verify approach. A solution that fluctuates around the targeted-inflation steady state is given by

$$\pi_t^{(*)} = \pi_*\gamma_*\exp\left[-\frac{1}{\psi}\sigma\epsilon_t\right],\quad \gamma_* = \exp\left[\frac{\sigma^2}{2(\psi-1)\psi^2}\right]. \tag{4}$$

We can also obtain a solution that fluctuates around the deflation steady state:

$$\pi_t^{(D)} = \pi_*\gamma_D\exp\left[-\frac{1}{\psi}\sigma\epsilon_t\right],\quad \gamma_D = \frac{1}{\pi_* r}\exp\left[-\frac{\sigma^2}{2\psi^2}\right]. \tag{5}$$

Figure 1: Inflation Dynamics in the Two-Equation Model



Notes: In the top left panel, the blue line shows the targeted-inflation equilibrium and the red line shows the deflation equilibrium. In the bottom panels, the shaded areas corresponds to periods in which the system is in the deflation regime. For the endogenous-sunspot equilibrium the regime switch is triggered by extreme shocks ϵ_t that exceed the thresholds depicted in the upper right panel by the dashed lines.

This solution differs from (4) only with respect to the constant γ_D, and has the same dynamics. We refer to $\pi_t^{(*)}$ as the targeted-inflation equilibrium and $\pi_t^{(D)}$ as the deflation equilibrium associated with (3).

In addition to the equilibria in (4) and (5) one can consider equilibria in which a two-state Markov-switching sunspot shock $s_t \in \{0, 1\}$ triggers moves from a targeted-inflation regime to a deflation regime and vice versa:

$$\pi_t^{(s)} = \pi_* \gamma(s_t) \exp \left[-\frac{1}{\psi} \sigma \epsilon_t \right]. \tag{6}$$

The constants $\gamma(0)$ and $\gamma(1)$ depend on the transition probabilities of the Markov switching process and ensure that (3) holds in every period t. The fluctuations of $\pi_t^{(s)}$ around $\pi_* \gamma(s_t)$ are identical to the fluctuations in the above targeted-inflation and deflation equilibria. The sunspot process could either evolve independently from the fundamental shock or it could be correlated with ϵ_t.[2] For instance, conditional on $s_{t-1} = 1$ (targeted-inflation regime), suppose that $s_t = 0$, i.e., the economy transitions to the deflation regime, if a large negative shock occurs: $\epsilon_t < \underline{\epsilon}_1$. Similarly, the economy exits the deflation regime, if a large positive shock occurs: $\epsilon_t > \underline{\epsilon}_2$.

A numerical illustration is provided in Figure 1. The upper-right panel depicts the evolution of the shock ϵ_t. The upper-left panel compares the paths of net inflation under the targeted-inflation equilibrium and the deflation equilibrium. The difference between the inflation paths is the level shift due to the constants γ_* versus γ_D. The bottom panel shows two sunspot equilibria with visible shifts from the targeted-inflation regime to the deflation regime (shaded areas) and back. In the left panel the sunspot evolves exogenously, whereas on the right it is endogenous in the sense that it gets triggered by extreme realizations of ϵ_t, which exceed the thresholds $\underline{\epsilon}_1$ and $\underline{\epsilon}_2$, respectively.

There exist many other solutions to (3). The local dynamics around the deflation steady state, ignoring the ZLB constraint, are indeterminate and it is possible to find alternative deflation equilibria. Moreover, Benhabib, Schmitt-Grohé, and Uribe (2001a) studied alternative equilibria in which the economy transitions from the targeted-inflation regime to a deflation regime and remains in the deflation regime permanently in continuous-time perfect foresight monetary models. Such equilibria can also be constructed in our model and one of them is discussed in more detail in the Online Appendix. In the remainder of this paper we will restrict our attention to equilibria of a New Keynesian DSGE model that are akin to $\pi_t^{(*)}$, $\pi_t^{(D)}$, and $\pi_t^{(s)}$ with an exogenously evolving sunspot shock.

[2]We thank Mike Woodford for the suggestion to explore equilibria in which the sunspot is triggered by fundamentals.

3 A Prototypical New Keynesian DSGE Model

The DSGE model we consider is the New Keynesian model studied in An and Schorfheide (2007). The model economy consists of perfectly competitive final-goods-producing firms, a continuum of monopolistically competitive intermediate goods producers, a continuum of identical households, and a government that engages in active monetary and passive fiscal policy. This model has been widely studied in the literature and many of its properties are discussed in the textbook by Woodford (2003). To keep the dimension of the state space manageable we abstract from capital accumulation and wage rigidities. We describe the preferences and technologies of the agents in Section 3.1, summarize the equilibrium conditions in Section 3.2, and characterize the steady states of the model in Section **??**.

3.1 Preferences and Technologies

Households. Households derive utility from consumption C_t relative to an exogenous habit stock and disutility from hours worked H_t. We assume that the habit stock is given by the level of technology A_t, which ensures that the economy evolves along a balanced growth path despite the quasi-linear preferences. We also assume that the households value transaction services from real money balances, detrended by A_t, and include them in the utility function. The households maximize

$$\mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s\left(\frac{(C_{t+s}/A_{t+s})^{1-\tau}-1}{1-\tau}-H_{t+s}+\chi V\left(\frac{M_{t+s}}{P_{t+s}A_{t+s}}\right)\right)\right], \tag{7}$$

subject to budget constraint

$$P_tC_t+T_t+M_t+B_t=P_tW_tH_t+M_{t-1}+R_{t-1}B_{t-1}+P_tD_t+P_tSC_t.$$

Here β is the discount factor, $1/\tau$ is the intertemporal elasticity of substitution, and P_t is the price of the final good. The households supply labor services to the firms, taking the real wage W_t as given. At the end of period t households hold money in the amount of M_t. They have access to a bond market where nominal government bonds B_t that pay gross interest R_t are traded. Furthermore, the households receive profits D_t from the firms and pay lump-sum taxes T_t. SC_t is the net cash inflow from trading a full set of state-contingent securities.

Real money balances enter the utility function in an additively separable fashion. An empirical justification of this assumption is provided by Ireland (2004). As a consequence, the equilibrium has a block diagonal structure under the interest-rate feedback rule that we will specify below: the level of output, inflation, and interest rates can be determined independently of the money stock. We assume that the marginal utility $V'(m)$ is decreasing in real money balances m and reaches zero for $m = \bar{m}$, which is the amount of money held in steady state by households if the net nominal interest rate is zero. Since the return on holding money is zero, it provides the rationale for the ZLB on nominal rates. The usual transversality condition on asset accumulation applies.

Firms. The final-goods producers aggregate intermediate goods, indexed by $j \in [0, 1]$, using the technology:

$$Y_t = \left(\int_0^1 Y_t(j)^{1-\nu} dj \right)^{\frac{1}{1-\nu}}.$$

The firms take input prices $P_t(j)$ and output prices P_t as given. Profit maximization implies that the demand for inputs is given by

$$Y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-1/\nu} Y_t.$$

Under the assumption of free entry into the final-goods market, profits are zero in equilibrium and the price of the aggregate good is given by

$$P_t = \left(\int_0^1 P_t(j)^{\frac{\nu-1}{\nu}} dj \right)^{\frac{\nu}{\nu-1}}. \tag{8}$$

We define inflation as $\pi_t = P_t / P_{t-1}$.

Intermediate good j is produced by a monopolist who has access to the following production technology:

$$Y_t(j) = A_t H_t(j), \tag{9}$$

where A_t is an exogenous productivity process that is common to all firms and $H_t(j)$ is the firm-specific labor input. Labor is hired in a perfectly competitive factor market at the real wage W_t. Intermediate-goods-producing firms face quadratic price adjustment costs of the form

$$AC_t(j) = \frac{\phi}{2}\left(\frac{P_t(j)}{P_{t-1}(j)} - \bar{\pi}\right)^2 Y_t(j),$$

where ϕ governs the price stickiness in the economy and $\bar{\pi}$ is a baseline rate of price change that does not require the payment of any adjustment costs. In our quantitative analysis we set $\bar{\pi} = 1$, that is, it is costless to keep prices constant. Firm j chooses its labor input $N_t(j)$ and the price $P_t(j)$ to maximize the present value of future profits

$$\mathbb{E}_t\left[\sum_{s=0}^{\infty} \beta^s Q_{t+s|t}\left(\frac{P_{t+s}(j)}{P_{t+s}} Y_{t+s}(j) - W_{t+s}H_{t+s} - AC_{t+s}\right)\right]. \tag{10}$$

Here, $Q_{t+s|t}$ is the time t value to the household of a unit of the consumption good in period $t+s$, which is treated as exogenous by the firm.

Government Policies. Monetary policy is described by an interest rate feedback rule of the form

$$R_t = \max\left\{1, \left[r\pi_*\left(\frac{\pi_t}{\pi_*}\right)^{\psi_1}\left(\frac{Y_t}{\gamma Y_{t-1}}\right)^{\psi_2}\right]^{1-\rho_R} R_{t-1}^{\rho_R} e^{\sigma_R \epsilon_{R,t}}\right\}. \tag{11}$$

Here r is the steady state real interest rate, π_* is the target-inflation rate, and $\epsilon_{R,t}$ is a monetary policy shock. The key departure from much of the New Keynesian DSGE literature is the use of the max operator to enforce the ZLB. Provided that the ZLB is not binding, the central bank reacts to deviations of inflation from the target rate π_* and deviations of output growth from γ.

The government consumes a stochastic fraction of aggregate output and government spending evolves according to

$$G_t = \left(1 - \frac{1}{g_t}\right) Y_t. \tag{12}$$

The government levies a lump-sum tax T_t (or provides a subsidy if T_t is negative) to finance any shortfalls in government revenues (or to rebate any surplus). Its budget constraint is given by

$$P_t G_t + M_{t-1} + R_{t-1} B_{t-1} = T_t + M_t + B_t. \tag{13}$$

Exogenous shocks. The model economy is perturbed by three exogenous processes. Aggregate productivity evolves according to

$$\ln A_t = \ln \gamma + \ln A_{t-1} + \ln z_t, \text{ where } \ln z_t = \rho_z \ln z_{t-1} + \sigma_z \epsilon_{z,t}. \tag{14}$$

Thus, on average the economy grows at the rate γ and z_t generates exogenous fluctuations of the technology growth rate. We assume that the government spending shock follows the AR(1) law of motion

$$\ln g_t = (1 - \rho_g) \ln g_* + \rho_g \ln g_{t-1} + \sigma_g \epsilon_{g,t}. \tag{15}$$

While we formally introduce the exogenous process g_t as a government spending shock, we interpret it more broadly as an exogenous demand shock that contributes to fluctuations in output. The monetary policy shock $\epsilon_{R,t}$ is assumed to be serially uncorrelated. We stack the three innovations into the vector $\epsilon_t = [\epsilon_{z,t}, \epsilon_{g,t}, \epsilon_{r,t}]'$ and assume that $\epsilon_t \sim iidN(0, I)$.

Unlike some of the other papers in the ZLB literature, e.g. Christiano, Eichenbaum, and Rebelo (2011) and Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2012), we do not include a discount factor shock in the model. We follow the strand of the literature that has estimated three-equation DSGE models that are driven by a technology shock, a demand (government spending), and a monetary policy shock and has documented that such models fit U.S. data for output growth, inflation, and interest rates reasonably well before the Great Recession. When we consider the sunspot equilibrium, we introduce another exogenous variable s_t, which follows a Markov-switching process

$$\mathbb{P}\{s_t = 1\} = \begin{cases} (1 - p_{00}) & \text{if } s_{t-1} = 0 \\ p_{11} & \text{if } s_{t-1} = 1 \end{cases} \tag{16}$$

3.2 Equilibrium Conditions

Since the exogenous productivity process has a stochastic trend, it is convenient to characterize the equilibrium conditions of the model economy in terms of detrended consumption and output: $c_t = C_t/A_t$ and $y_t = Y_t/A_t$. The consumption Euler equation is given by

$$1 = \beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\tau} \frac{1}{\gamma z_{t+1}} \frac{R_t}{\pi_{t+1}} \right]. \tag{17}$$

We define

$$\mathcal{E}_t = \mathbb{E}_t\left[\frac{c_{t+1}^{-\tau}}{\gamma z_{t+1}\pi_{t+1}}\right], \tag{18}$$

which will be useful in the computational algorithm. In a symmetric equilibrium in which all firms set the same price $P_t(j)$, the price-setting decision of the firms leads to the condition

$$\begin{aligned} 1 \;=\; & \frac{1}{\nu}(1-c_t^{\tau}) + \phi(\pi_t - \bar{\pi})\left[\left(1-\frac{1}{2\nu}\right)\pi_t + \frac{\bar{\pi}}{2\nu}\right] \\ & -\phi\beta\mathbb{E}_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\tau}\frac{y_{t+1}}{y_t}(\pi_{t+1}-\bar{\pi})\pi_{t+1}\right] \end{aligned} \tag{19}$$

The aggregate resource constraint can be expressed as

$$c_t = \left[\frac{1}{g_t} - \frac{\phi}{2}(\pi_t - \bar{\pi})^2\right] y_t. \tag{20}$$

It reflects both government spending as well as the resource cost (in terms of output) caused by price changes. Finally, we reproduce the monetary policy rule

$$R_t = \max\left\{1, \left[r\pi_*\left(\frac{\pi_t}{\pi_*}\right)^{\psi_1}\left(\frac{y_t}{y_{t-1}}z_t\right)^{\psi_2}\right]^{1-\rho_R} R_{t-1}^{\rho_R} e^{\sigma_R \epsilon_{R,t}}\right\}. \tag{21}$$

We do not use a measure of money in our empirical analysis and therefore drop the equilibrium condition that determines money demand.

As the two-equation model in Section 2, the New Keynesian model with the ZLB constraint has two steady states, which we refer to as the targeted-inflation and the deflation steady state. In the targeted-inflation steady state inflation equals π_* and the gross interest rate equals $r\pi_*$, while in the deflation steady state inflation equals $1/r$ and the interest rate equals one.

4 Solving the Model Subject to the ZLB Constraint

We now discuss some key features of the algorithm that is used to solve the nonlinear DSGE model presented the previous section. Additional details can be found in the Online Appendix. We compute three of the equilibria that we studied in the context of the two-equation model presented in Section 2: a targeted-inflation equilibrium in which the economy fluctuates around the targeted-inflation steady state; a minimal-state-variable deflation equilibrium with fluctuations near the deflation steady state; and a sunspot equilibrium with a targeted-inflation and a deflation regime. While the targeted-inflation equilibrium has been computed by Judd, Maliar, and Maliar (2010), Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2012) and Gust, Lopez-Salido, and Smith (2012) for similar DSGE models, to the best of our knowledge this is the first paper to compute the deflation and the sunspot equilibria for a discrete-time DSGE model with a full set of stochastic shocks.

The minimum set of state variables associated with our DSGE models is $(R_{t-1}, y_{t-1}, g_t, z_t, \epsilon_{R,t})$ and the sunspot variable s_t, where applicable, which we collectively label as $\mathcal{S}_t$. An (approximate) solution of the DSGE model is a set of decision rules

$$\pi_t = \pi(\mathcal{S}_t; \Theta),\ \mathcal{E}_t = \mathcal{E}(\mathcal{S}_t; \Theta),\ c_t = c(\mathcal{S}_t; \Theta),\ y_t = y(\mathcal{S}_t; \Theta),\ \text{and } R_t = R(\mathcal{S}_t; \Theta)$$

that solves the nonlinear rational expectations system

$$\xi\left(c_t, \pi_t, y_t\right) = \phi\beta I\!\!E_t\left[(c_{t+1})^{-\tau} y_{t+1}(\pi_{t+1} - \bar{\pi})\pi_{t+1}\right] \tag{22}$$

$$c_t^{-\tau} = \beta R_t \mathcal{E}_t \tag{23}$$

$$y_t = \left[\frac{1}{g_t} - \frac{\phi}{2}\left(\pi_t - \bar{\pi}\right)^2\right]^{-1} c_t \tag{24}$$

$$R_t = \max\left\{1, \left[r_*\pi_*\left(\frac{\pi_t}{\pi_*}\right)^{\psi_1}\left(\frac{y_t}{y_{t-1}} z_t\right)^{\psi_2}\right]^{1-\rho_R} R_{t-1}^{\rho_R} e^{\epsilon_{R,t}}\right\} \tag{25}$$

where $\mathcal{E}_t$ was defined in (18) and $\xi(.)$ is defined as

$$\xi\left(c, \pi, y\right) = c^{-\tau} y\left\{\frac{1}{\nu}\left(1 - c^{\tau}\right) + \phi(\pi - \bar{\pi})\left[\left(1 - \frac{1}{2\nu}\right)\pi + \frac{\bar{\pi}}{2\nu}\right] - 1\right\}. \tag{26}$$

We utilize a global approximation using fourth-order Chebyshev polynomials following Judd (1992). The solution algorithm amounts to specifying a grid of points $\mathcal{G} = \{\mathcal{S}_1, \ldots, \mathcal{S}_M\}$ in the model's state space and solving for the vector Θ such that the sum of squared residuals associated with (18) and (22) are minimized for $\mathcal{S}_t \in \mathcal{G}$. Note that conditional on $\pi(\mathcal{S}_t; \Theta)$ and $\mathcal{E}(\mathcal{S}_t; \Theta)$ the Equations (23)-(25) determine $c(\mathcal{S}_t; \Theta)$, $y(\mathcal{S}_t; \Theta)$, and $R(\mathcal{S}_t; \Theta)$ and therefore these equations hold exactly. There are two non-standard aspects of our solution method that we will now discuss in more detail: a piecewise smooth representation of the functions $\pi(\cdot;\Theta)$ and $\mathcal{E}(\cdot;\Theta)$ and the iterative procedure of choosing grid points $\mathcal{G}$.

We show in Section B of the Online Appendix that the solution to a simplified linearized version of our DSGE model entails piece-wise linear decision rules. While Chebyshev polynomials, which are smooth functions of the states, can in principle approximate functions with a kink, such approximations are quite inaccurate for low-order polynomials. Thus, unlike Judd, Maliar, and Maliar (2010), Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2012) and Gust, Lopez-Salido, and Smith (2012), we use a piece-wise smooth approximation of the functions $\pi(\mathcal{S}_t)$ and $\mathcal{E}(\mathcal{S}_t)$ by postulating

$$\begin{aligned}\pi(\mathcal{S}_t;\Theta) &= \zeta_t f^1_\pi(\mathcal{S}_t;\Theta) + (1-\zeta_t) f^2_\pi(\mathcal{S}_t;\Theta) \\ \mathcal{E}(\mathcal{S}_t;\Theta) &= \zeta_t f^1_{\mathcal{E}}(\mathcal{S}_t;\Theta) + (1-\zeta_t) f^1_{\mathcal{E}}(\mathcal{S}_t;\Theta),\end{aligned} \tag{27}$$

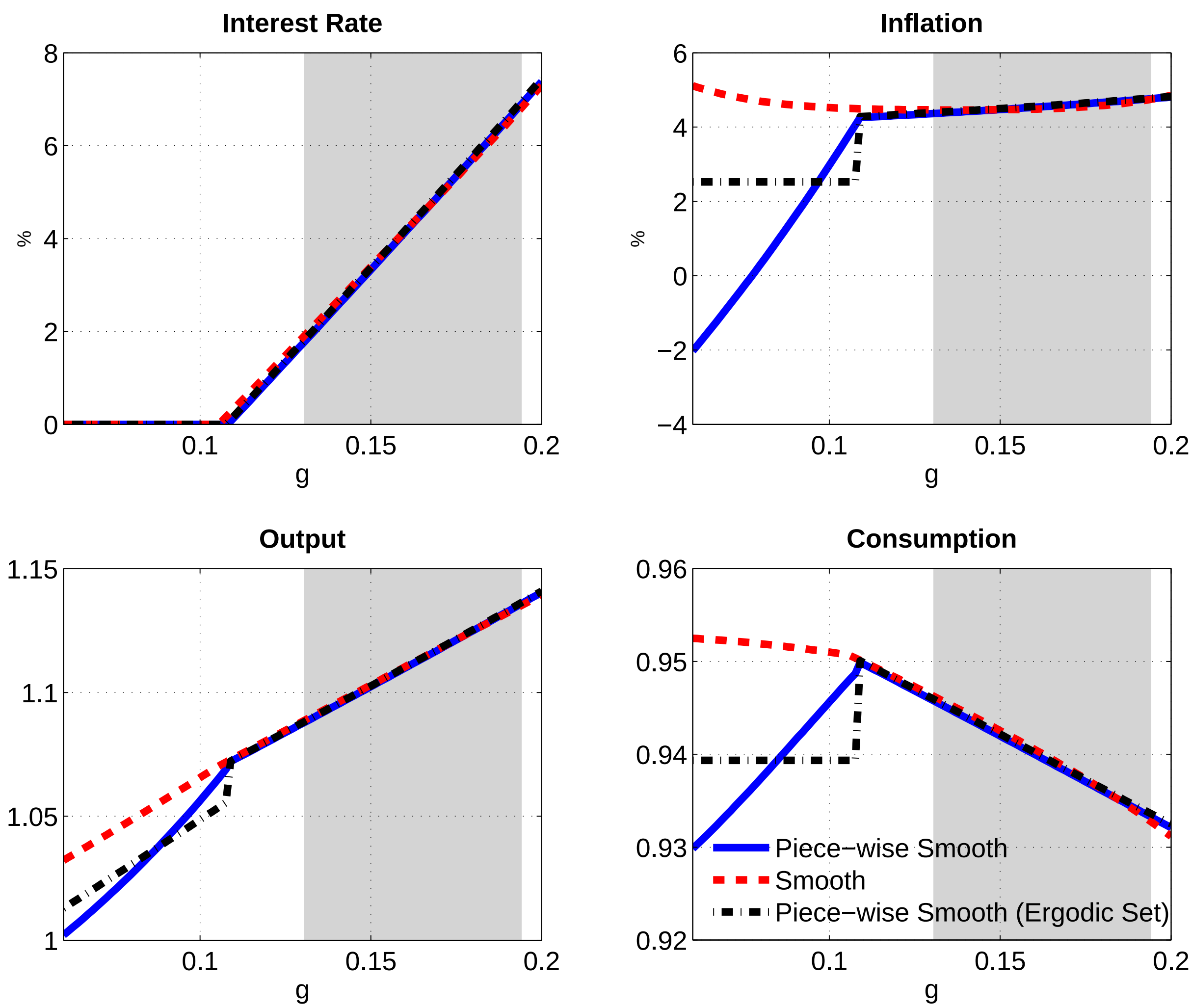
where $\zeta_t = I\{R(\mathcal{S}_t;\Theta) > 1\}$ is an indicator that shows the ZLB is slack. The functions f^i_j are linear combinations of a complete set of Chebyshev polynomials up to fourth order, where the weights are given by a vector Θ. Our method is flexible enough to allow for a kink in all decision rules and not just R_t which has a kink by its construction.[3]

In our experience the flexibility of the piece-wise smooth approximation yields more accurate decision rules, especially for inflation. Figure 2 shows a slice of the decision rules where we set $R_{t-1} = 1$, $y_{t-1} = y_*$, $z = 0$ and $\epsilon_{R,t} = 0$ and vary g_t in a wide range. The solid blue decision rules are based on the piece-wise smooth approximation in (27) whereas the dashed red decision rules are obtained using a single set of Chebyshev polynomials. When approximated smoothly, the decision rules fail to capture the kinks that are apparent in the piece-wise smooth approximation. For instance, the decision rule for output illustrates that the (marginal) government-spending multiplier is sensitive to the ZLB - it is noticeably larger in the area of the state space where the ZLB binds – which is not captured by the smooth approximation.

Projection methods that require the solution to be accurate on a fixed grid, e.g. a tensor product grid, become exceedingly difficult to implement as the number of state variables

[3]This method would work well if instead of kinks we had differentiable decision rules whose slopes were changing rapidly in the neighborhood of where the ZLB becomes slack.

Figure 2: Sample Decision Rules



Notes: The gray shading show 95% coverage of the ergodic distribution in the targeted-inflation equilibrium.

increases above three. While the Smolyak grid proposed by Krueger and Kubler (2004) can alleviate the curse of dimensionality to some extent, we build on recent work by Judd, Maliar, and Maliar (2010) and use stochastic simulations to generate a grid that adapts to the ergodic distribution of the model. However, unlike Judd, Maliar, and Maliar (2010) we need the model solution to be accurate not only for high-density points of the model-implied ergodic distribution of $\mathcal{S}_t$, but also for values of the state variables that can explain the recession of 2008-09. It turns out that these values are several standard deviations away from the center of the ergodic distribution. Thus, we combine draws from the ergodic distribution with states that are extracted from data on output growth, inflation, and interest rates to generate the grid $\mathcal{G}$. This ensures that our approximation remains accurate in the area of the state space that is relevant for the empirical analysis.

The gray shading in Figure 2 shows the 95% coverage of the ergodic distribution. While values of g to the left of the shaded area are unlikely under the ergodic distribution, they turn out to be important to explain the observations in 2008-09. The economy reaches the ZLB for values of g that are smaller than 0.11. The dashed-dotted black decision rules are computed based on a grid that consists only of points from the ergodic distribution, whereas the solid blue decision rules are obtained using a grid that mixes points from the ergodic distribution and the filtered states. Under the ergodic-set grid the solution algorithm essentially does not evaluate the nonlinear rational expectations system for states that imply a binding ZLB. Since we use a piece-wise smooth approximation, the decision rules for inflation and consumption are constant as a function of g for low values, reflecting steady states and the particular initialization of Θ.

Recall that the model solution is obtained by solving for the vector Θ that minimizes the sum of squared residuals associated with (18) and (22) for $\mathcal{S}_t \in \mathcal{G}$. The multiplicity of equilibria in our DSGE model implies that this minimization has multiple solutions, i.e. the objective function is multi-modal. To find the targeted-inflation equilibrium we start from a log-linear approximation around the targeted-inflation steady state that ignores the ZLB. This log-linear approximation is used to initialize Θ in the determination of the decision rules and to generate the first set of draws from the ergodic distribution of $\mathcal{S}_t$. We then apply a clustering algorithm to obtain 130 grid points for a solution based on fourth-order Chebychev polynomials. Using this solution we simulate a new set of points from the ergodic distribution and apply the filtering algorithm described in Section 5.3 to extract 51 state vectors from observations on output, inflation, and interest rates for 2000:Q1 to 2012:Q3. We combine these 51 filtered states with grid points from the simulation and re-solve the model. These steps are repeated five more times until the decision rules stabilize.

To solve for the deflation equilibrium, we initialize Θ using constant decision rules for π_t and $\mathcal{E}_t$ that capture the deflation steady state. For the initial grid we generate draws from the ergodic distribution using these decision rules. As we did for the targeted-inflation equilibrium, we proceed iteratively by simulating states from the ergodic distribution to obtain grid points to construct the next solution. For the deflation equilibrium, the grid is generated by retaining every M'th state from the simulated sequence, where M is chosen to obtain 130 grid points. Under the deflation equilibrium nominal interest rates are zero with substantial probability. Using this time-separated grid algorithm ensures that the fraction of grid points associated with a nominal interest rate of zero is accurately captured in our solution grid. This is important for obtaining decision rules that are accurate near the ZLB where the system spends a lot of time. We do not use filtered states for the grid because it is difficult to rationalize U.S. data using the deflation equilibrium (see Section 5.3 for more details).

For the sunspot equilibrium, we need to compute two sets of decision rules, one for $s_t = 0$ and another one for $s_t = 1$. We use the decision rules from the targeted-inflation equilibrium and the deflation equilibrium to initialize the regime-specific decision rules for the sunspot equilibrium. The solution is constructed iteratively. As for the targeted-inflation equilibrium, we use a grid that contains states simulated from the ergodic distribution and states that are obtained by filtering.

5 Quantitative Analysis

The quantitative analysis consists of four parts. In Section 5.1 we estimate the parameters of the DSGE model under the assumption that the economy was in the targeted-inflation equilibrium from 1984 to 2007. These parameter estimates are the starting point for the subsequent analysis. In Section 5.2 we compare the ergodic distribution of inflation and interest rates under the three equilibria. In Section 5.3 we use the model to estimate a sequence of historical states for the period 2000:Q1 to 2012:Q3. Conditional on the estimated states during the Great Recession of 2008-09, Section 5.4 assesses the effect of fiscal and monetary policy interventions in the targeted-inflation and the sunspot equilibria.

Table 1: DSGE Model Parameters

$\tau = 1.50$	$r = 1.0070$	$\gamma = 1.0048$	$\nu = 0.1$
$g_* = 1/0.85$	$\phi = 75.75$	$\pi_* = 1.0063$	$\bar{\pi} = 1$
$\psi_1 = 1.36$	$\psi_2 = 0.80$	$\rho_R = 0.65$	$\sigma_R = 0.0021$
$\rho_g = 0.86$	$\sigma_g = 0.0078$	$\rho_Z = 0.11$	$\sigma_z = 0.0103$

5.1 Estimation under Targeted-Inflation Equilibrium

The parameter values for the subsequent analysis are obtained by estimating the DSGE model described in Section 3 under the assumption that the economy is in the targeted-inflation equilibrium from 1984:Q1 to 2007:Q4 using output growth, inflation, and interest rate data. Since the ZLB was not binding during this period, we replace the global approximation discussed in Section 4 by a second-order perturbation approximation. In the area of the state-space that is empirically relevant for our estimation sample, the decision rules obtained under these two solution methods are virtually identical. The data for the estimation was extracted from the FRB St. Louis FRED database (November 2012 vintage). Output growth is defined as real GDP (GDPC96) growth converted into per capita terms. Our measure of population is Civilian Noninstitutional Population (CNP16OV). We compute population growth rates as log differences and apply an eight-quarter backward-looking moving average filter to the growth rates to smooth out abrupt changes in the population growth series. Inflation is defined as the log difference in the GDP deflator (GDPDEF) and the interest rate is the average effective federal funds rate (FEDFUNDS) within each quarter. We use Bayesian techniques described in detail in An and Schorfheide (2007) and report posterior mean estimates in Table 1. These estimates are in line with estimates for New Keynesian DSGE models that have been reported elsewhere in the literature. A detailed description of the prior distribution underlying this estimation is provided in Appendix D.

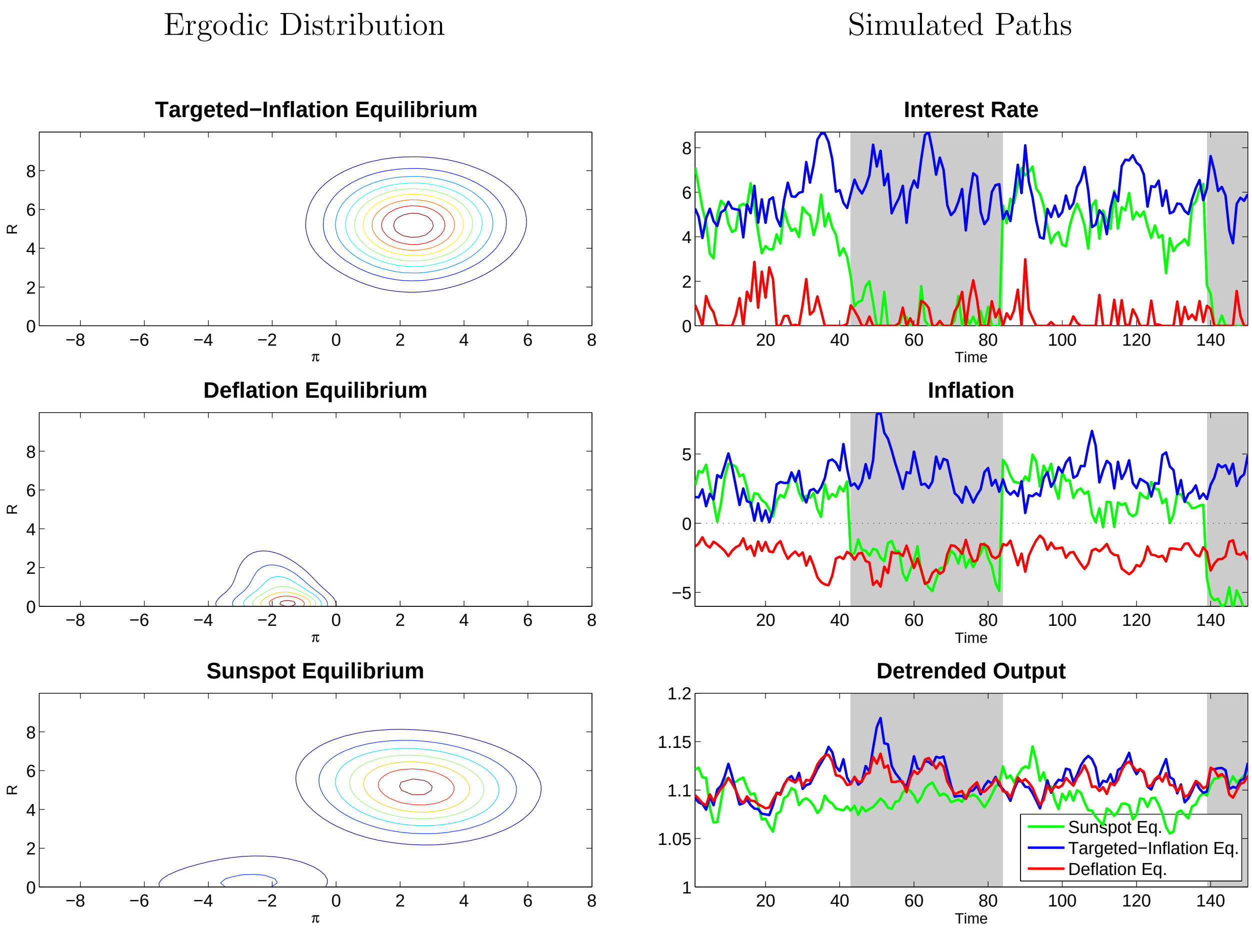
For the sunspot equilibrium we also need to specify values for the transition probabilities p_{00} and p_{11}. These parameters determine the expected durations of staying in each regime. Since there is no clear empirical observation to identify the transition probabilities, we informally chose $p_{00} = 0.95$ and $p_{11} = 0.99$. These values make the deflation regime ($s_t = 0$) less persistent than the targeted-inflation regime ($s_t = 1$). Since the agents' decision rules under the targeted-inflation regime are influenced by the potential switch to the deflation regime, it turns out that the average inflation and interest rates conditional on the regime $s_t = 1$ do not coincide with the targeted-inflation steady state. To ensure that the targeted-inflation regime in the sunspot equilibrium can explain the pre-2007 data as well as the targeted-inflation equilibrium, we change π_* from the value reported in Table 1 to $\pi_* = 1.0071$, which corresponds to an annualized inflation target of 2.85%.

5.2 Equilibrium Dynamics

In order to evaluate the ergodic distributions associated with the targeted-inflation, the deflation, and the sunspot equilibria, we simulate a long sequence of draws from each of the equilibria. The left column of Figure 3 depicts contour plots of the ergodic distributions. The ergodic distribution of the targeted-inflation equilibrium is approximately centered at the steady state values, which are 2.5% inflation and an interest rate of 5.3% annually. The contours for the deflation equilibrium are concentrated near the ZLB and peak at an inflation rate of about -1.9%. This inflation rate is larger than the steady state value of -2.8%. By construction, the sunspot equilibrium generates a bimodal ergodic distribution of inflation and interest rates. However, this bimodal distribution is not simply a mixture of the distributions associated with the target-inflation and the deflation equilibria. Since agents expect regime changes to occur in the future the decision rules in the two regimes of the sunspot equilibrium are different from the decision rules in the pure equilibria.

Given our parameter estimates the probability of hitting the ZLB under the targeted-inflation equilibrium is essentially zero. This is not surprising since the estimation sample ranges from 1984 to 2007, which is a period of above-zero interest rates and low macroeconomic volatility. In the deflation regime the interest rate frequently hits the ZLB and may stay at zero for multiple periods. While inflation is mostly positive in the targeted-inflation regime, it is always negative in the deflation regime.

Figure 3: Ergodic Distribution and Simulated Paths



Notes: Left column depicts contour plots of the joint probability density function (kernel density estimate) of interest rates and inflation. Interest rate and inflation are net rates at an annual rate.

The right column of Figure 3 shows simulated paths of interest rates, inflation rates, and output growth for the three equilibria, using the same shock innovations. The shaded areas correspond to periods in which the deflation regime is active in the sunspot equilibrium. The simulated paths from the sunspot equilibrium alternate between periods of low interest rates coupled with deflation and periods of high interest and inflation rates. The regime switch induced by the sunspot shock triggers a strong adjustment of the nominal variables. The time paths of output growth are very similar in the targeted-inflation and deflation equilibrium, and it appears to be slightly lower in the sunspot equilibrium.[4] The correlation of (detrended) output and inflation have different signs in different equilibria. In the targeted-inflation equilibrium and the targeted-inflation regime of the sunspot equilibrium, this correlation is positive (0.77 and 0.73, respectively), and it is negative in the deflation equilibrium and the deflationary regime of the sunspot equilibrium (-0.84 and -0.94, respectively). This result is similar to the findings of Eggertsson (2009) and Mertens and Ravn (2013) who argue that the aggregate demand curve may become upward sloping in the deflation regime and thus demand shocks may lead to a negative comovement of prices and output.

The focus of this paper is not normative but it is worth mentioning that there is nothing necessarily "bad" about the deflation equilibrium: the distance between actual and desired inflation (0%) is roughly the same in the deflation and the targeted-inflation equilibria (resulting in similar adjustment costs) and average consumption and its volatility is roughly the same. In fact, since the interest rate is closer to the Friedman rule of 0%, the welfare loss due to holding money is actually smaller in the deflation equilibrium. We leave a full-blown normative analysis along the lines of Aruoba and Schorfheide (2011) to future work.

5.3 Extracting Historical Shocks

We now use the DSGE model to determine the sequence of shocks that lead to the Great Recession in 2008-09 and subsequent period where the ZLB continues to bind. The filtered state variables for 2009:Q1 will provide the initial conditions for the policy experiments in Section 5.4. We extract two sequences of shocks and states: one sequence is obtained under the assumption that the U.S. economy was in the targeted-inflation equilibrium, whereas the other sequence was obtained assuming that the sunspot equilibrium prevailed since 2000:Q1.[5]

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[4]To provide more details, the ZLB binds 40% of the time in the deflation equilibrium, 10% of the time in the sunspot equilibrium (always when $s_t = 0$ and never in the targeted-inflation equilibrium. Annualized output growth is 1.93% in both the targeted-inflation equilibrium and the deflation equilibrium and it is 1.87% in the sunspot equilibrium.

[5]We initialized the filter in 2000:Q1 to make sure that the initialization does not affect inference for the states in 2009:Q1.

Because the U.S. has never experienced a prolonged period of deflation since 1960, the deflation equilibrium is empirically implausible and not considered in the subsequent analysis.

The DSGE model can be represented as a state space model. Let y_t be the 3×1 vector of observables consisting of output growth, inflation, and nominal interest rates. The vector x_t stacks the continuous state variables which are given by $x_t = [R_t, y_t, y_{t-1}, z_t, g_t, A_t]'$ and $s_t \in \{0, 1\}$ is the Markov-switching process.

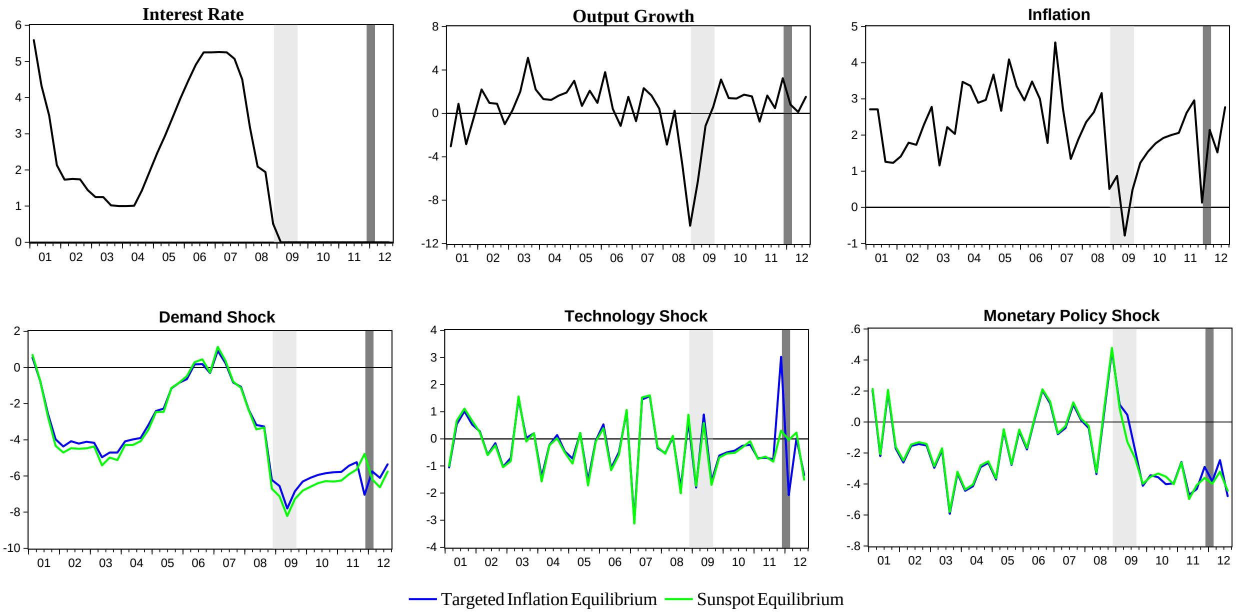
$$\begin{aligned} y_t &= \Psi(x_t) + \nu_t \\ \mathbb{P}\{s_t = 1\} &= \begin{cases} (1 - p_{00}) & \text{if } s_{t-1} = 0 \\ p_{11} & \text{if } s_{t-1} = 1 \end{cases} \\ x_t &= F_{s_t}(x_{t-1}, \epsilon_t) \end{aligned} \tag{28}$$

The first equation in (28) is the measurement equation, where $\nu_t \sim N(0, \Sigma_\nu)$ is a vector of measurement errors. The second equation represents law of motion of the Markov-switching process. The third equation corresponds to the law of motion of the continuous state variables. The vector $\epsilon_t \sim N(0, I)$ stacks the innovations $\epsilon_{z,t}$, $\epsilon_{g,t}$, and $\epsilon_{R,t}$. The functions $F_0(\cdot)$ and $F_1(\cdot)$ are generated by the model solution procedure. Under the targeted-inflation equilibrium the state-transition equation $x_t = F(x_{t-1}, \epsilon_t)$ is time-invariant and the Markov switching process s_t does not affect outcomes. The state vector x_t is extracted from the observables using a particle filter, also known as sequential Monte Carlo filter.[6]

Figure 4 depicts the data described in Section 5.1 (top row) and the time-path of shocks extracted conditional on the two equilibria (bottom row) over the period 2000:Q1-2012:Q2. Interest rates have been essentially zero since 2009, output growth fell considerably in the second half of 2008 and GDP deflator inflation was below 50bp (annualized) at the beginning of 2009 as well as in late 2011. Gray shading shows periods when the probability of sunspot

[6] Gordon and Salmond (1993) and Kitagawa (1996) made early contributions to the development of particle filters. In the economics literature the particle filter has been applied to analyze stochastic volatility models, e.g., Pitt and Shephard (1999), and nonlinear DSGE models following Fernández-Villaverde and Rubio-Ramírez (2007). Surveys of sequential Monte Carlo filtering are provided, for instance, in the engineering literature by Arulampalam, Maskell, Gordon, and Clapp (2002) and in the econometrics literature by Giordani, Pitt, and Kohn (2011). A detailed description of the particle filter used in the subsequent quantitative analysis is provided in the Online Appendix.

Figure 4: Data and Extracted Shocks



Notes: The top row shows the data used in filtering where all variables are in annualized percentage units. The bottom row shows the extracted shocks as percentage deviations from their respective steady states.

equilibrium being in the deflation regime is greater than zero, where the larger the probability, the darker the shading.

Before the Great Recession, the sunspot equilibrium is in the targeted-inflation regime and the two equilibria require essentially identical shocks to explain the data. This period is characterized by demand shocks that are persistently below their mean between 2001 and 2005, and expansionary monetary policy especially between 2003 and 2006. The demand shock starts to drop below its mean in 2007 and continues to do so at the beginning of 2009:Q1. At the end of 2009:Q1, our particle filter shows the first indication that a sunspot switch, with a probability of 28% in 2009:Q2 and 10% in 2009:Q3. The reason for this is apparent in the ergodic distribution presented in Figure 3. In the targeted-inflation equilibrium, having observations where interest rate is zero and inflation is very small, or negative is an extremely rare event, while it is very likely in the sunspot equilibrium, as long as the equilibrium switches to the deflation regime. In fact, the only reason the probability of a regime switch remain low is because inflation does not go down enough – in 2009:Q2 it falls to -0.7% only to go up to 0.5% in the next quarter. In 2011:Q4, when inflation suddenly falls to 0.2% while growth remains at 3.1%, the particle filter signals essentially 100% probability of a switch to the deflation regime as the only way to explain low inflation and high growth in the targeted-inflation regime is a very large technology shock. What is also noteworthy about the period after 2008 is the persistent negative monetary policy shocks from 2009:Q3 onwards that average -1.8 standard deviations. Keeping in mind that the monetary policy shock has an iid stochastic process, this shows a substantial intervention by the Federal Reserve. From the extracted shocks, it is quite clear that the data prefers the targeted-inflation equilibrium (or equivalently the targeted-inflation regime of the sunspot equilibrium) as an explanation for the period 2008-2010. We turn to discussing our justification of considering the deflation regime of the sunspot equilibrium in our policy analysis in Section 5.4.1.

5.4 Policy Experiments

In this section our main goal is to highlight the differences of the targeted-inflation equilibrium and the sunspot equilibrium in terms the response of the economy to policy interventions. The recent literature has emphasized that the effects of expansionary fiscal policies on output may be larger if the economy is at or near the ZLB. In the absence of the ZLB a typical interest rate feedback rule implies that the central bank raises nominal interest rates in response to rising inflation and output caused by an increase in government spending. This monetary contraction raises the real interest rate, reduces private consumption, and overall dampens the stimulating effect of the fiscal expansion. If the economy is at the ZLB, the expansionary fiscal policy is less likely to be accompanied by a rise in interest rates because the feedback portion of the policy rule tends to predict negative interest rates. Without a rising nominal interest rate, the increase in inflation that results from the fiscal expansion reduces the real rate. In turn, current-period demand is stimulated, amplifying the positive effect on output. In fact, Figure 2 shows that when the ZLB starts to bind, the response of output to an increase in government spending is larger, and that consumption goes up.

Unlike what was mostly the standard practice before the Great Recession, we cannot simply plot impulse responses due to the highly non-linear nature of our environment: we need to specify the initial conditions – where the economy is just before the intervention –

and the size of the intervention.[7] We use the Great Recession and the subsequent period in the U.S. as our laboratory and calibrate our policy intervention to a portion of the ARRA of February 2009, along with an expansionary monetary policy by the Federal Reserve. This enables us to also discuss how this particular policy affected the U.S. economy. We do this from an ex-ante perspective – exactly how an impulse-response analysis would do it – but also from an ex-post perspective, since we have extracted the historical shocks and are able to consider counterfactuals.

5.4.1 Empirical Relevance versus Policy Relevance

Before we turn to the results, we want to respond to a question that may arise at this point. Our results in Section 5.3 show that ex post, the evidence in favor of a shift to the deflation regime during the period 2009-2010 is weak. As such, the sunspot equilibrium, and any possible differences in policy responses relative to the targeted-inflation equilibrium, may appear not to be empirically relevant. However, this does not mean it is not policy relevant: the possibility of the U.S. economy being in what we call a sunspot equilibrium and experiencing a switch to the deflation regime was very much considered in the monetary policy debates at the time. This is highlighted by the following quote from an article by the President of the Federal Reserve Bank of St. Louis, James Bullard in Bullard (2010):

> During this recovery, the U.S. economy is susceptible to negative shocks that may dampen inflation expectations. This could push the economy into an unintended, low nominal interest rate steady state. Escape from such an outcome is problematic. (...) The United States is closer to a Japanese-style outcome today than at any time in recent history. (...)

[7]The main source of nonlinearity is the occasionally-binding nature of ZLB. Without it, given parameters estimated using data before the Great Recession, the environment will be only mildly nonlinear, and a linear approximation would largely suffice. In that case, impulse-responses can be obtained analytically given the linear approximation, are invariant to initial conditions and scale up and down linearly with the size of shocks.

> Promising to remain at zero for a long time is a double-edged sword. The policy is consistent with the idea that inflation and inflation expectations should rise in response to the promise and that this will eventually lead the economy back toward the targeted equilibrium. But the policy is also consistent with the idea that inflation and inflation expectations will instead fall and that the economy will settle in the neighborhood of the unintended steady state, as Japan has in recent years.

Here, James Bullard is talking about various shocks, some of which may possibly be actions or announcements by the Federal Reserve, leading the economy to settle near the deflation steady state. In our model these kind of shocks are captured by a switch of the sunspot process to the deflation regime. This is direct evidence that our deflation regime was a concern for policy makers.

One of the key characteristics of the deflation regime in the sunspot equilibrium is the sharp decline in inflation to possibly deflationary levels. As Figure 4 shows, inflation starts to decline from about 2.5% in the Summer of 2008 to 0.5% in 2008:Q4. When asked in February 2009, the month in which ARRA was signed into law, the participants of the Survey of Professional Forecasters (SPF) replied that there is a 12% probability for 2009 and a 8% probability for 2010 that annual inflation will be less than zero (deflation). These numbers are at least an order of magnitude larger than the historical averages[8] and consistent with the anticipation of a switch to a deflation regime.

We use the evidence presented in this section in two ways. First, we think it provides sufficient justification for considering the sunspot equilibrium as a relevant alternative alongside the targeted-inflation equilibrium. Second, we set $s_{2009:Q2} = 0$ in our policy experiments, that is, we consider what a policy-maker who believes that a sunspot switch has occurred would see about the consequences of its actions in 2009:Q2.

[8]The average of first-quarter responses to the same question between 1992 and 2008 is 0.7% probability for the current year and 0.9% for the following year.

5.4.2 Details of the Policy Experiments

Taking the filtered states from 2009:Q1 ($t = T_*$) and $s_{2009:Q2} = 0$ in the sunspot equilibrium as given, we now study the effects of policy interventions during the Great Recession, starting in 2009:Q2 and lasting for eight quarters. In particular we consider an increase in government spending that is potentially combined with an expansionary monetary policy. To make the experiment more realistic, the fiscal policy intervention is calibrated to a portion of the ARRA of February 2009. ARRA consisted of a combination of tax cuts and benefits; entitlement programs; and funding for federal contracts, grants, and loans. We focus on the third component because it can be interpreted as an increase in g_t. We model the ARRA spending as a one-period shock δ^{ARRA} to the demand shock process, where $\delta^{ARRA} = 0.011$, which is roughly $1.4\sigma_g$. Since $\hat{g}_t$ is serially correlated the effect of the shock in the h'th period on the level of $\hat{g}_t$ is given by $\rho_g^{h-1}\delta^{ARRA}$. In Section F of the Online Appendix we describe how we use data on the disbursement of ARRA funds to determine δ^{ARRA} for our model. While the actual path of the received funds is not perfectly monotone, the calibrated intervention in the DSGE model roughly matches the actual intervention both in terms of magnitude and decay rate. In our empirical analysis we consider a pure fiscal policy intervention as well as a combination of expansionary fiscal and monetary policy.

To assess the ex-ante predicted effect of an increase in government spending we simulate the model economy forward with and without policy intervention. Along both paths we set the monetary policy shocks $\epsilon_{R.t}$ equal to zero and simulate the technology shocks z_t. Along the baseline path the demand shock evolves according to

$$\hat{g}_t = \rho_g \hat{g}_{t-1} + \sigma_g \epsilon_{g,t},$$

whereas along the post-intervention path the demand shock is given by

$$\hat{g}_t^I = \hat{g}_t + \rho_g^{t-T_*-1}\delta^{ARRA}. \tag{29}$$

and we define the difference $X^I - X$ to be the effect of the intervention, where X^I is a generic variable as simulated with the intervention and X is the baseline path of the same variable.

Since a fiscal expansion creates an upward pressure on the nominal interest rates via the feedback mechanism of the interest rate rule, in principle there is scope for amplifying the effect of the fiscal stimulus by a monetary policy that keeps interest rates at or near zero. Thus, we also consider a combination of expansionary fiscal and monetary policy, where the central bank intervention is implemented using a sequence of unanticipated monetary policy shocks $\epsilon_{R,t}$.[9] To avoid implausibly large interventions, we choose these shocks such that they are no larger than two standard deviations in absolute value, and the interest-rate intervention is no larger than one percentage point in annualized terms in any quarter. Thus, we implicitly assume that the FOMC would renege on a policy to keep interest rates near zero for an extended period of time in states of the world in which output growth and or inflation turn out to be high.

For the ex-post policy analysis we use the particle filter to obtain estimates of the exogenous shock processes for the period 2009:Q2-2011:Q1.[10] Since the actual path of the demand shock already contains the effect of fiscal expansion due to ARRA, we define the counterfactual path as

$$\hat{g}^C_{t|t} = \hat{g}_{t|t} - \rho_g^{t-T_*-1}\delta^{ARRA}, \tag{30}$$

where $\hat{g}_{t|t}$ denotes the filtered demand shock. To measure the effect of the combined fiscal and monetary policy we also set the filtered monetary policy shocks to zero (which turn out to be negative past 2009:Q2) when computing the counterfactual outcomes. The ex-post effect of the intervention is defined as $X^{observed} - X^C$ where $X^{observed}$ is the observed value of a generic variable and X^C is the counterfactual path along which the policy intervention is removed. Details on the algorithms to compute the effects of the policy interventions are reported in Appendix C.

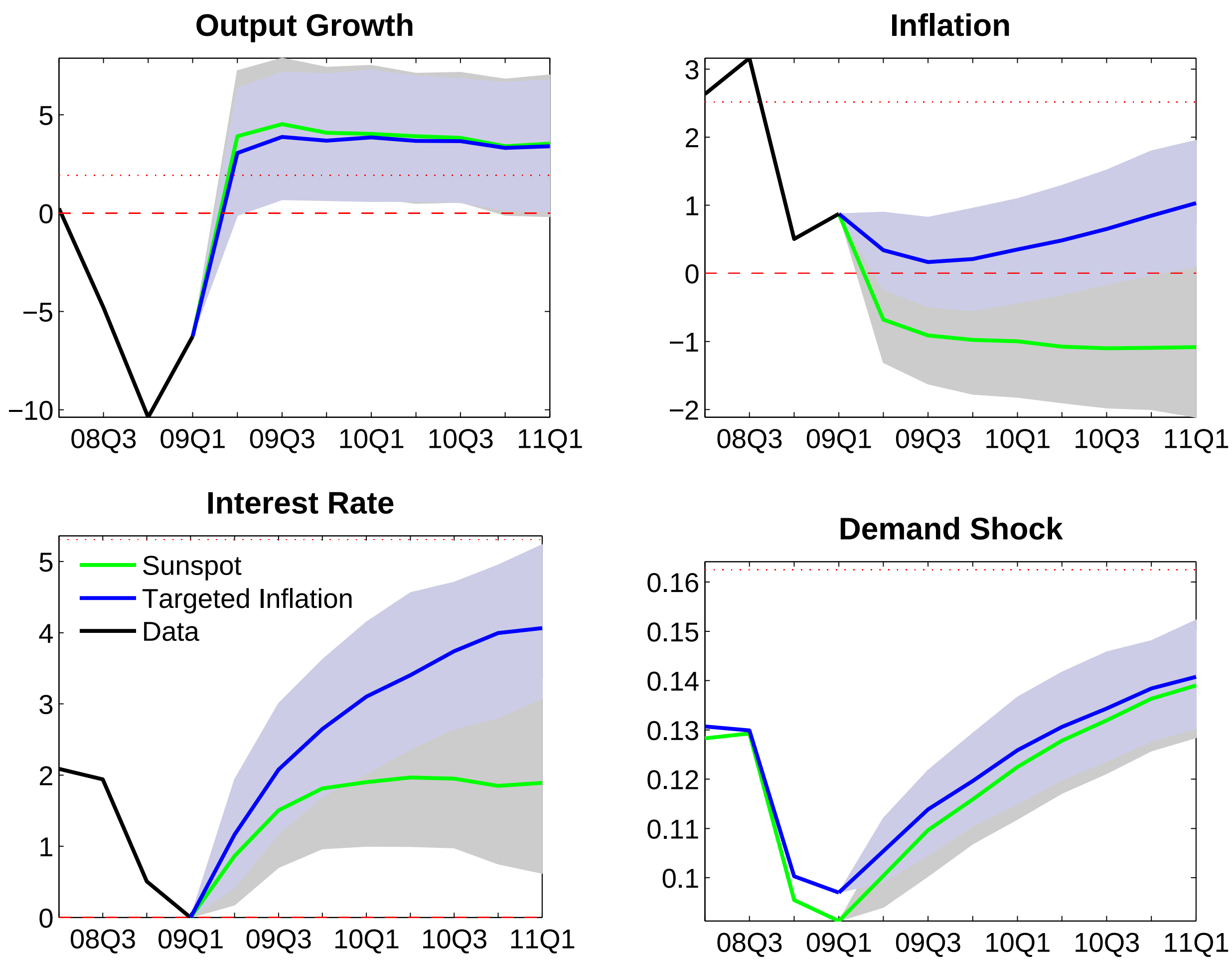
5.4.3 No Intervention

In order to understand the effects of the policy intervention, it is instructive to first look at what was labeled as X in the discussion in the previous section: the expected evolution of

[9] A detailed discussion about the advantages and disadvantages of using unanticipated versus anticipated monetary policy shocks to generate predictions conditional on an interest rate path is provided in Del Negro and Schorfheide (2012).

[10] Since we use $s_{2009:Q2} = 0$ in the sunspot equilibrium, we run the particle filter conditional on this. This yields filtered states that are slightly different from those presented in Figure 4.

Figure 5: No Intervention



Notes: Figure shows data (black) and the results of simulations with no intervention from targeted-inflation equilibrium (blue) and sunspot equilibrium (green): pointwise medians (solid); 20%-80% percentiles (shaded area). All variables except for the demand shock are expressed in terms of annualized percentage units. The dashed red lines show the unconditional mean for these variables.

the U.S. economy absent a policy intervention. As of 2009:Q1, output growth is at −6%, inflation is at 0.9% and the interest rate had hit the ZLB. Conditioning on the states in Figure 4 that deliver these observations, and in the case of the sunspot equilibrium, conditional also on $s_{2009:Q2} = 0$, we simulate the model forward for eight quarters. Figure 5 shows the paths of output growth, inflation and interest rate from 2009:Q2 through 2011:Q1, along with the data for the previous four quarters for comparison. The bottom right panel shows the demand shock as extracted by the particle filter through 2009:Q1 and the simulated paths conditional on the value on 2009:Q1.

The demand shock is more than 40% below its steady state value in 2009:Q1 and as a result, when simulated forward we see a sustained increase. In the targeted-inflation equilibrium, this causes a sharp jump in output growth initially and higher-than-average growth for the rest of the periods. The central bank reacts to this sharp increase by increasing the interest rate immediately. This also keeps the inflation rate under control – in over 80% of simulations it remains under 2% by the end of the simulation. In the sunspot equilibrium, output growth behaves similarly but since in 2009:Q2 we assume a switch to the deflation equilibrium, the inflation rate remains in negative territory for much of the simulations.[11] To reiterate, the scenario James Bullard and the SPF respondents were considering (deflation for an extended period) is a very real possibility in the sunspot equilibrium (over 80% probability) but it is only a remote probability in the targeted-inflation equilibrium.

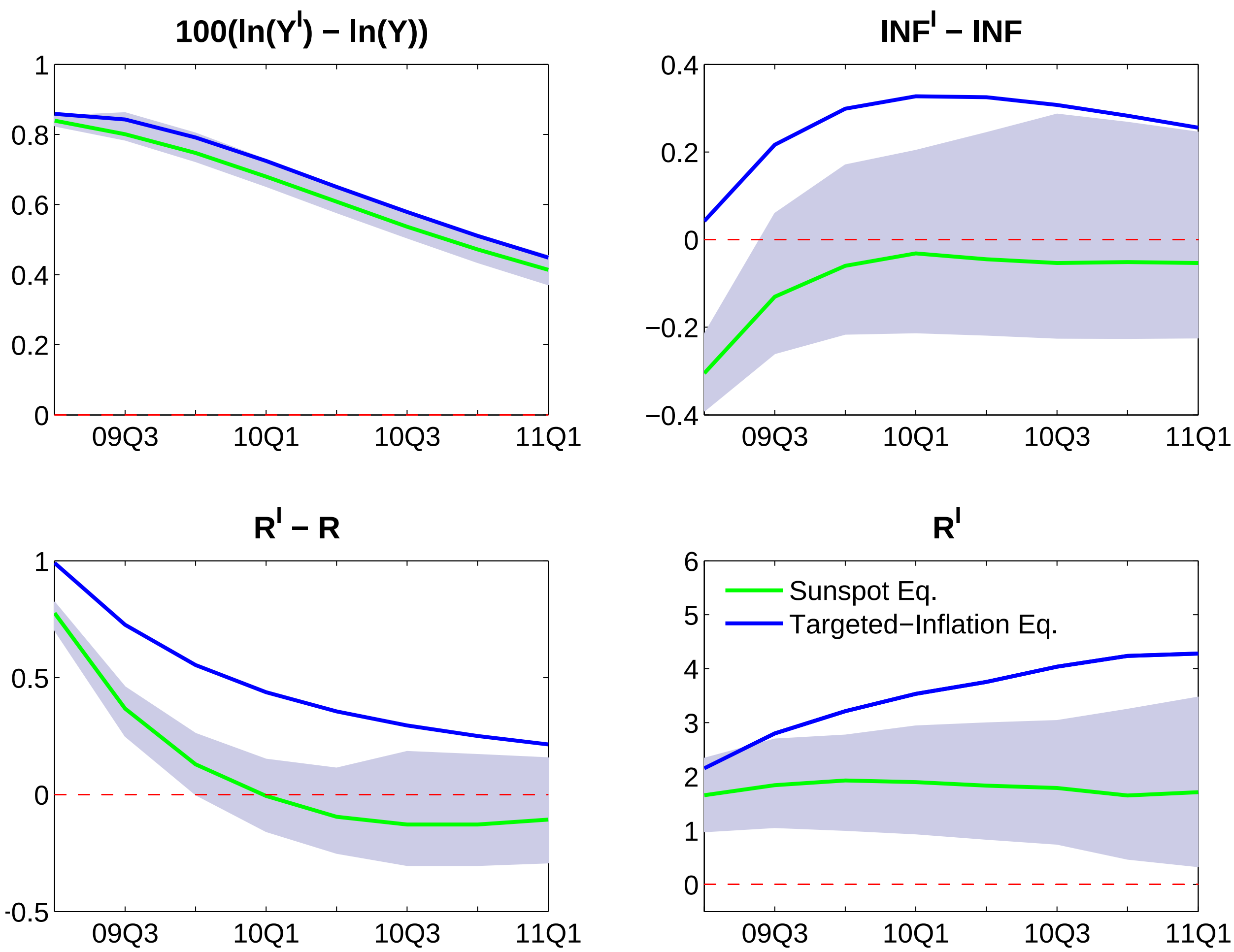
5.4.4 Fiscal Policy Intervention (Ex Ante)

We first look at the effect of a fiscal policy intervention, introduced in 2009:Q2, from an ex-ante perspective. Figure 6 overlays the effects of the fiscal expansion for the targeted-inflation equilibrium and the sunspot equilibrium. The figure shows (pointwise) median responses as well as upper and lower 20% percentiles of the distribution of the intervention effects $X^I - X$ for the sunspot equilibrium. The effects in the targeted-inflation equilibrium mirror a "standard" response to a government spending shock in a New Keynesian DSGE model. Both output and inflation increase, and in response the central bank raises interest rates. Output increases by 80bp and monotonically reverts back to the no-intervention level, whereas the response of inflation is hump-shaped and peaks at about 30bp.[12] It's important to emphasize that even though the economy is at the ZLB in 2009:Q1, as shown in Figure 5 it is expected to leave it immediately due to the rapid increase in demand, even in the absence of the intervention. With the intervention, the interest rate increases even faster.

[11] In the simulations we also simulate the sunspot variable and allow for a switch from the deflation regime to the targeted-inflation regime according to its assumed law of motion. The bands for the sunspot equilibrium are larger since in some paths the economy switches to the latter regime and it behaves differently.

[12] Since the nonlinearities under the targeted-inflation equilibrium are weak, the bands that characterize the distribution of responses are very narrow and thus are not shown.

Figure 6: Fiscal Policy Intervention in Targeted-Inflation and Sunspot Equilibrium



Notes: Figure compares intervention effects from targeted-inflation equilibrium (blue) and sunspot equilibrium (green): pointwise medians (solid); 20%-80% percentiles for sunspot equilibrium (shaded area). Inflation and the interest rate are expressed in terms of annualized percentage rates.

Since we are conditioning on $s_{2009:Q2} = 0$ for the sunspot equilibrium, agents expect the economy to be in the deflation regime with high probability for the subsequent periods. As we showed in Section 5.2, inflation and output dynamics change drastically in the deflation regime. In particular, a demand shock may move output and inflation in opposite directions, leading to a fall in inflation as a result of the fiscal intervention we consider here. Indeed Figure 6 shows that inflation falls by about 30bp on impact. As time passes, the band for inflation widens substantially since in some paths the sunspot variable switches to the targeted-inflation regime and the inflation response becomes more "standard". Interestingly,

Table 2: Multipliers

Intervention	Targeted-Inflation			Sunspot			
	1Q	4Q	8Q	s_t Path	1Q	4Q	8Q
Ex Ante Policy Analysis – Conditional on 2009:Q1 States							
Fiscal	0.80	0.92	0.97	0xxxxxxx	0.80	0.88	0.93
Fiscal + Monetary	1.24	1.69	2.09	0xxxxxxx	1.06	1.32	1.52
Ex Post Policy Analysis – Conditional on 2009:Q2 - 2011:Q1 Shocks							
Fiscal	1.33	1.26	1.22	01111111	1.17	1.20	1.18
				00000000	1.17	1.09	1.08
Fiscal + Monetary	1.33	1.62	2.03	01111111	1.17	1.57	1.99
				00000000	1.17	1.23	1.34

the output response is about the same as the one in the targeted-inflation equilibrium. There are three forces at work here. First is the direct effect of government spending, which increases output. Second, because inflation falls on impact and the economy is expected to remain at or near the ZLB, the real interest rate goes up and as a result consumption falls, reducing output. Third, since output growth is above the target γ, and inflation is below the target π_*, the central bank does not increase the interest rate as much as it does in the targeted-inflation equilibrium, and this gives an additional boost to output. In net, the total increase ends up being around 80bp on impact.

Based on the simulations we can calculate government spending multipliers that measure the effect of the fiscal intervention on output relative to the overall increase in government spending. We consider a multiplier defined as

$$\mu = \frac{\sum_{\tau=1}^{H}(Y_\tau^I - Y_\tau)}{\sum_{\tau=1}^{H}(G_\tau^I - G_\tau)}.$$

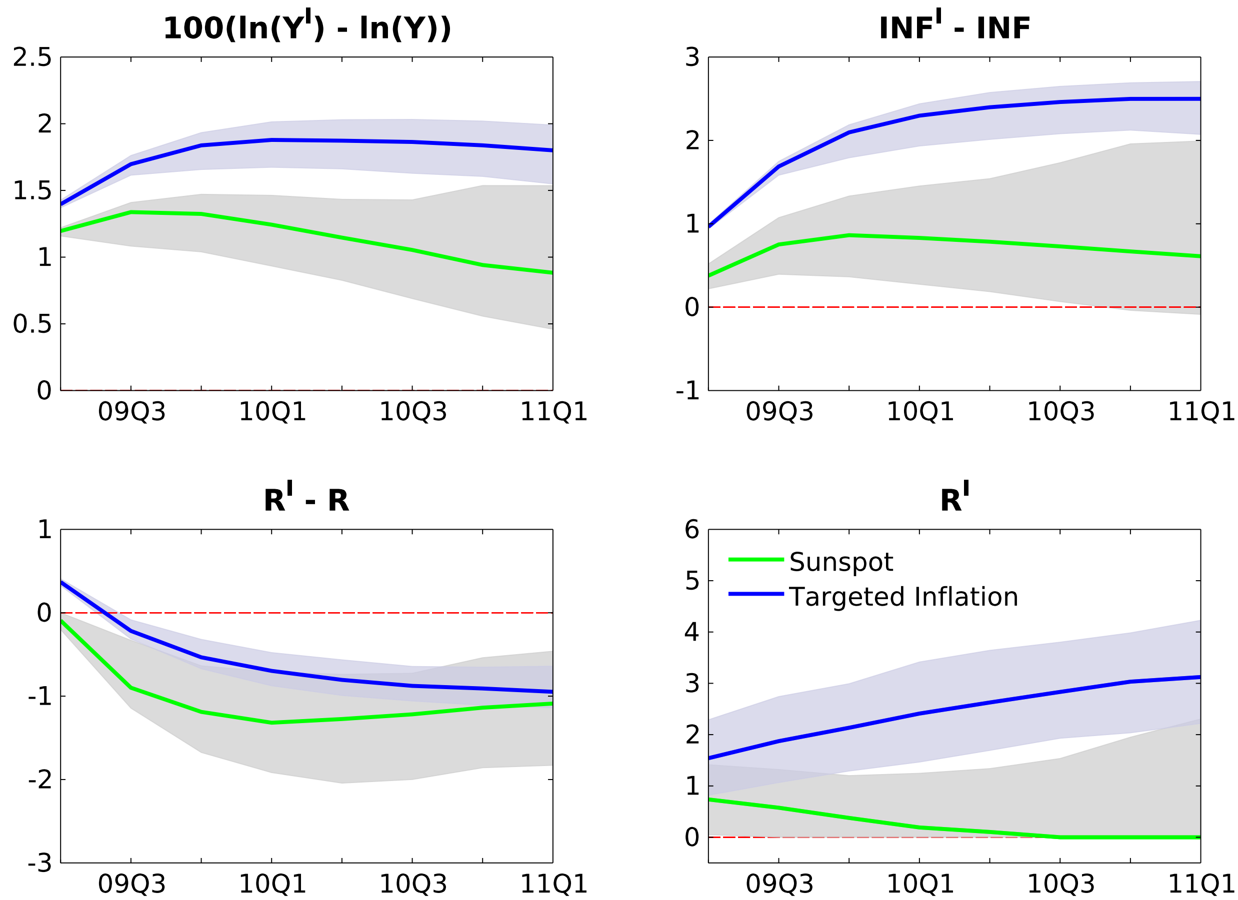
Our measure is cumulative over the lifetime of the intervention and the means across simulations are tabulated for various types of policy interventions in Table 2. The multipliers for the ex-ante policy exercise underlying Figure 6 are reported in the first row of the table, labeled "Fiscal." Under the targeted-inflation equilibrium the multipliers range from 0.80 ($H = 1$) to 0.97 ($H = 8$). For the sunspot equilibrium the multipliers are slightly smaller, ranging from 0.80 to 0.93. These multipliers are substantially smaller than those reported in the literature for fiscal interventions at the ZLB. This is because the extra private consumption channel does not kick in as the economy leaves the ZLB instantly.

5.4.5 Combined Fiscal and Monetary Policy Intervention (Ex Ante)

The results for the combined intervention is shown in Figure 7. Since the interest rate tends to increase rapidly in the targeted-inflation equilibrium, there is a lot of scope for monetary policy interventions because the ZLB poses hardly a constraint. The monetary policy shock hits the two-standard-deviation bound we place on the intervention in every period at the median response. This very large monetary intervention leads to a 1.5% increase in output and an almost 1% increase in inflation on impact, both of which are substantially larger than those for just a fiscal intervention. As the intervention continues, inflation and output continue to increase. The multipliers for this combined policy are 1.24 on impact and 2.09 after eight quarters, which are 55% and 115% larger than the multipliers with just the fiscal intervention. Thus, we can conclude that monetary policy provides a very large additional boost to the fiscal intervention under the targeted-inflation equilibrium.

Turning to the sunspot equilibrium, the reduction in the interest rate generated by the combined policy intervention is even more sizable. As a result, along many paths, the economy hits the ZLB and by the sixth period more than half of the paths have the interest rate at the ZLB. Despite the large reduction in the interest rate, the output reaction is smaller, relative to the targeted-inflation equilibrium. As we explained in Section 5.4.4, a major difference between the two equilibria is their reaction to demand shocks: in the sunspot equilibrium, output and inflation move in opposite directions and the reduction in inflation as a result of a fiscal expansion would increase the real interest rate, if the nominal interest rate is near zero. In this case, since the economy is pushed to the ZLB by the monetary policy, this channel becomes quite strong and dampens the output reaction. The multipliers, reported in Table 2 for this combined policy are 1.06 on impact and 1.52 after eight quarters, which are only 33% and 63% larger than the multipliers with just the fiscal

Figure 7: Both Policies in Targeted-Inflation and Sunspot Equilibrium



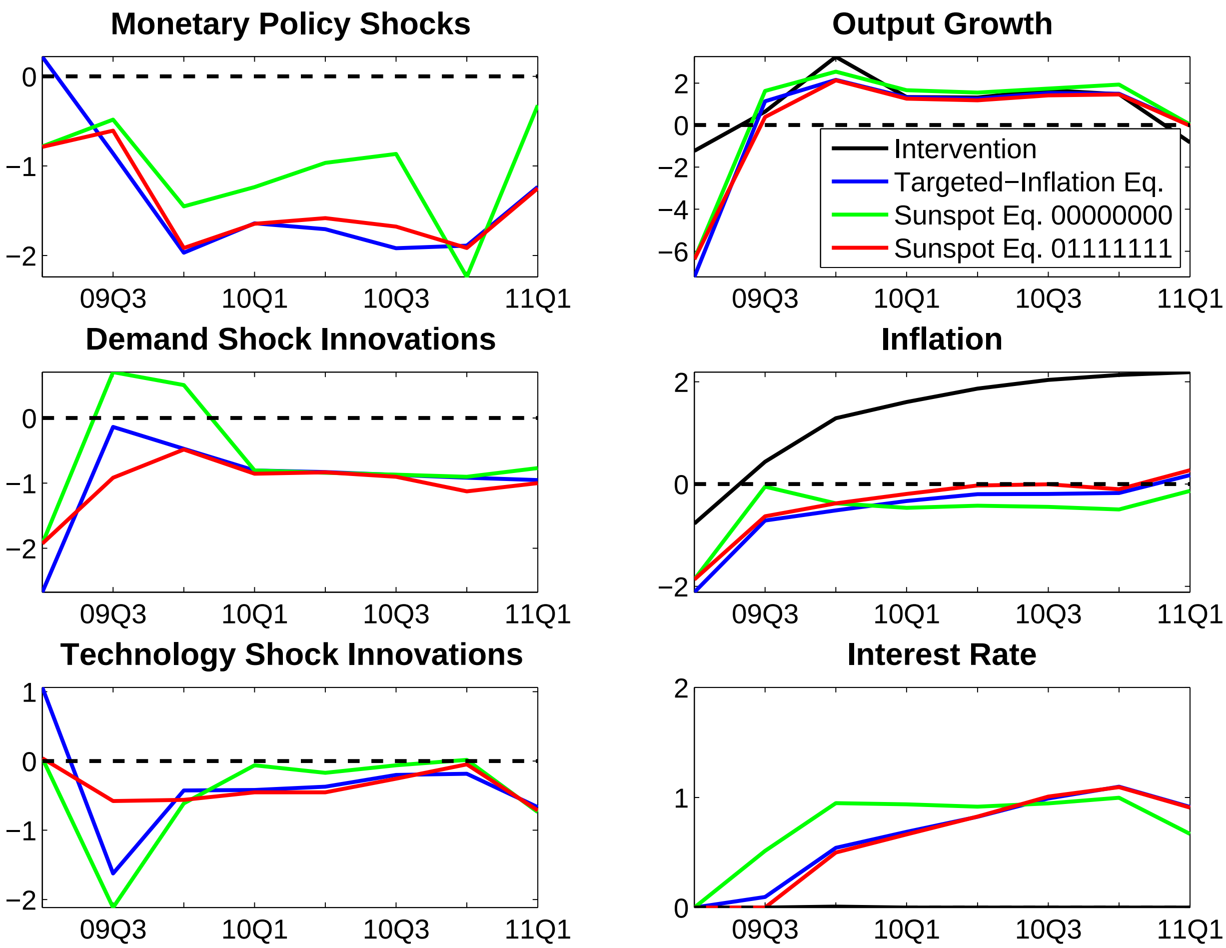
Notes: Figure compares intervention effects from targeted-inflation equilibrium (blue) and sunspot equilibrium (green): pointwise medians (solid); 20%-80% percentiles (shaded area). Inflation and the interest rate are expressed in terms of annualized percentage rates.

intervention. This means the additional boost due to monetary policy is only half as large in the sunspot equilibrium relative to the targeted-inflation equilibrium.

5.4.6 Ex-Post Policy Analysis

So far, we took an ex-ante perspective in that we simulated the path of the exogenous shocks from 2009:Q1 onwards. Now, we will take an ex-post perspective and condition on the filtered path of the exogenous processes. This is a straightforward exercise for the targeted-inflation equilibrium, but we need to take a stand on the path of the sunspot variable in the sunspot equilibrium. We already assumed $s_{2009:Q2} = 0$. We consider two alternatives. First, labeled

Figure 8: Ex-Post Policy Analysis in Targeted-Inflation and Sunspot Equilibrium



Notes: The first column of plots depicts filtered innovations for the monetary policy, the demand, and the technology shocks. The second column of plots depicts the actual paths of GDP growth, inflation, and interest rates (black) and three counterfactual paths (blue, green, and red) along which we set the filtered monetary policy shocks to zero and lower the innovation to the government spending shock in 2009:Q2 by the size of the ARRA intervention. Inflation and the interest rate are expressed in terms of annualized percentage rates.

00000000, we assume that the U.S. economy stays in the deflation regime for the duration of the exercise. Given that $p_{00} = 0.95$, this is a very likely outcome ex ante. Second, labeled 01111111, we assume that economy switches to the targeted-inflation regime in 2009:Q3 and stays there. We run the particle filter again to extract the states conditional on these paths of the sunspot variable.

Figure 8 shows the results. The panels on the left depict the extracted filtered shock innovations, expressed in multiples of their respective standard deviations. Two things are noteworthy. First, in 2009:Q2, the first period of the intervention, all three models show a large negative demand shock, in the order of -2 standard deviations. In the absence of the ARRA stimulus the shock would have been -3.5 standard deviations or worse. Second, throughout much of this two-year period, there were large negative monetary policy shocks, indicating that the Federal Reserve was engaging in active expansionary policy. The panels on the right side of Figure 8 show the paths of output growth, inflation and the interest rate under policy intervention and under three counterfactuals with no policy intervention.[13] Two important results emerge from all three counterfactuals. First, output would have fallen at an annualized rate by over 6% (instead of about 1.5%) in 2009:Q2 had ARRA not been enacted. Second, there would have been deflation for almost the entire two-year period. This is partially because absent the monetary intervention, interest rates would have hovered around 1%, generating downward pressure on prices.

The ex-post multipliers are tabulated in the bottom panel of Table 2. Under the scenario labeled "Fiscal" we simply remove the ARRA intervention from the filtered $\hat{g}_t$ process. Under the targeted-inflation equilibrium, this leads to ex-post multipliers ranging from 1.33 (1 quarter) to 1.22 (8 quarters). The multipliers for the 01111111 path of the sunspot equilibrium are close to these numbers as well while for the 00000000 path, they are somewhat smaller. The ex-post multipliers are larger than the ex-ante multipliers, because ex post the economy spends more time than expected at the ZLB, due to adverse demand and expansionary monetary policy shocks. However, our multipliers are considerably smaller than those obtained by Christiano, Eichenbaum, and Rebelo (2011).

Turning to the effect of monetary policy, monetary policy had no extra boost in 2009:Q2 under all of the three scenarios because the economy is taken to the ZLB by an adverse demand shock. By the end of the two-year period, however, had the Fed not engaged in the expansionary policy indicated by the filtered states, then things would have been more dire. In terms of multipliers, the monetary intervention provided a boost of 67%, with a multiplier of 2.03, which is lower than the 115% expected ex-ante but still sizeable. The

[13] By construction our model delivers the data exactly when fed with the extracted filtered shocks with one exception: the sunspot equilibrium with the path 00000000 cannot deliver the positive inflation values observed after 2009:Q2.

numbers are similar, not surprisingly, in the 0111111 path of the sunspot equilibrium but if the realization of the sunspot equilibrium was given by the 00000000 path, then the boost due to monetary policy would have been only 24% with a combined multiplier of just 1.34.

5.4.7 Summary

We motivated the comparison of policy effects under the targeted-inflation and the sunspot equilibria using a quote from James Bullard that highlights policy makers' concerns in 2009 about a prolonged switch to a deflation regime. Indeed, based on our empirical analysis both the targeted-inflation equilibrium and the sunspot equilibrium with a switch from the targeted-inflation to the deflation regime were empirically plausible in the beginning of 2009.

We showed that from an ex-ante perspective the effects of fiscal policy on output under the two equilibria are quite similar, albeit for different economic reasons. However, we also found that the scope for expansionary monetary policy in conjunction with the fiscal stimulus is substantially smaller in the sunspot equilibrium because the economy tends to spend more time near the ZLB, and the government spending shock tends to lower rather than raise inflation rates.

Ex post, it turned out that the empirical evidence for an extended deflation regime under the sunspot equilibrium is very weak and that the targeted-inflation equilibrium or the sunspot equilibrium with a very short-lived switch to the deflation regime provide more plausible characterizations of the period from 2009 to 2010. The main conclusion from the ex-post analysis is that because the economy spent more time at the ZLB than predicted ex ante, fiscal policy was more effective and monetary policy was less effective.

6 Conclusion

We solve a New Keynesian DSGE model subject to a ZLB constraint on nominal interest rates, considering three equilibria: the standard targeted-inflation equilibrium, a minimal-state-variable deflation equilibrium, and a sunspot equilibrium. These equilibria differ in terms of their properties, especially near the ZLB and also in terms of the effectiveness of policy. We motivated the comparison of policy effects under the targeted-inflation and the sunspot equilibria using a quote from James Bullard that highlights policy makers' concerns in 2009 about a prolonged switch to a deflation regime.

We showed that from an ex-ante perspective the effects of fiscal policy on output under the two equilibria are quite similar, albeit for different economic reasons. However, we also found that the scope for expansionary monetary policy in conjunction with the fiscal stimulus is substantially smaller in the sunspot equilibrium because the economy tends to spend more time near the ZLB, and the government spending shock tends to lower rather than raise inflation rates. Ex post, it turned out that the empirical evidence for an extended deflation regime under the sunspot equilibrium is very weak and that the targeted-inflation equilibrium or the sunspot equilibrium with a very short-lived switch to the deflation regime provide more plausible characterizations of the period from 2009 to 2010. The main conclusion from the ex-post analysis is that because the economy spent more time at the ZLB than predicted ex ante, fiscal policy was more effective and monetary policy was less effective.

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Appendix to "Macroeconomic Dynamics Near the ZLB: A Tale of Two Equilibria"

A Solving the Two-Equation Model

The model is characterized by the nonlinear difference equation

$$\mathbb{E}_t[\pi_{t+1}] = \max\left\{\frac{1}{r},\ \pi_*\left(\frac{\pi_t}{\pi_*}\right)^{\psi}\exp[\epsilon_t]\right\}. \tag{A.1}$$

We assume that $r\pi_* \geq 1$ and $\psi > 1$.

The Targeted-Inflation Equilibrium and Deflation Equilibrium. Consider a solution to (A.1) that takes the following form

$$\pi_t = \pi_*\gamma\exp[\lambda\epsilon_t]. \tag{A.2}$$

We now determine values of γ and λ such that (A.1) is satisfied. We begin by calculating the following expectation

$$\begin{aligned}
\mathbb{E}_t[\pi_{t+1}] &= \pi_*\gamma\frac{1}{\sqrt{2\pi\sigma^2}}\int\exp[\lambda\epsilon]\exp\left[-\frac{1}{2\sigma^2}\epsilon^2\right]d\epsilon \\
&= \pi_*\gamma\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[\frac{1}{2}\lambda^2\sigma^2\right]\int\exp\left[-\frac{1}{2\sigma^2}(\epsilon-\lambda\sigma^2)^2\right]d\epsilon \\
&= \pi_*\gamma\exp\left[\frac{1}{2}\lambda^2\sigma^2\right].
\end{aligned}$$

Combining this expression with (A.1) yields

$$\gamma\exp[\lambda^2\sigma^2/2] = \max\left\{\frac{1}{r\pi_*},\ \gamma^{\psi}\exp[(\psi\lambda+1)\epsilon_t]\right\}. \tag{A.3}$$

By choosing $\lambda = -1/\psi$ we ensure that the right-hand-side of (A.3) is always constant. Thus, (A.3) reduces to

$$\gamma\exp[\sigma^2/(2\psi^2)] = \max\left\{\frac{1}{r\pi_*},\ \gamma^{\psi}\right\} \tag{A.4}$$

Depending on whether the nominal interest rate is at the ZLB ($R_t = 1$) or not, we obtain two solutions for γ by equating the left-hand-side of (A.4) with either the first or the second term in the max operator:

$$\gamma_D = \frac{1}{r\pi_*}\exp\left[-\frac{\sigma^2}{2\psi^2}\right] \quad \text{and} \quad \gamma_* = \exp\left[\frac{\sigma^2}{2(\psi-1)\psi^2}\right]. \tag{A.5}$$

The derivation is completed by noting that

$$\begin{aligned}\gamma_D^\psi &= \frac{1}{r\pi_*}\exp\left[-\frac{\sigma^2}{2\psi}\right] \leq \frac{1}{r\pi_*} \\ \gamma_*^\psi &= \exp\left[\frac{\sigma^2}{2(\psi-1)\psi}\right] \geq 1 \geq \frac{1}{r\pi_*}.\end{aligned}$$

A Sunspot Equilibrium. Let $s_t \in \{0,1\}$ denote the Markov-switching sunspot process. Assume that the system is in the targeted-inflation regime if $s_t = 1$ and that it is in the deflation regime if $s_t = 0$ (the 0 is used to indicate that the system is near the ZLB). The probabilities of staying in state 0 and 1, respectively, are denoted by ψ_{00} and ψ_{11}. We conjecture that the inflation dynamics follow the process

$$\pi_t^{(s)} = \pi_*\gamma(s_t)\exp[-\epsilon_t/\psi] \tag{A.6}$$

In this case condition (A.4) turns into

$$\begin{aligned}\mathbb{E}_t[\pi_{t+1}|s_t = 0]/\pi_* &= \big(\psi_{00}\gamma(0) + (1-\psi_{00})\gamma(1)\big)\exp[\sigma^2/(2\psi^2)] = \frac{1}{r\pi_*} \\ \mathbb{E}_t[\pi_{t+1}|s_t = 1]/\pi_* &= \big(\psi_{11}\gamma(1) + (1-\psi_{11})\gamma(0)\big)\exp[\sigma^2/(2\psi^2)] = [\gamma(1)]^\psi.\end{aligned}$$

This system of two equations can be solved for $\gamma(0)$ and $\gamma(1)$ as a function of the Markov-transition probabilities ψ_{00} and ψ_{11}. Then (A.6) is a stable solution of (A.1) provided that

$$[\gamma(0)]^\psi \leq \frac{1}{r\pi_*} \quad \text{and} \quad [\gamma(1)]^\psi \geq \frac{1}{r\pi_*}.$$

Sunspot Shock is Correlated with Fundamentals. As before, let $s_t \in \{0,1\}$ be a Markov-switching sunspot process. However, now assume that a state transition is triggered by certain realizations of the monetary policy shock ϵ_t. In particular, if $s_t = 0$, then suppose $s_{t+1} = 0$ whenever $\epsilon_{t+1} \leq \underline{\epsilon}_0$, such that

$$\psi_{00} = \Phi(\underline{\epsilon}_0),$$

where $\Phi(\cdot)$ is the cumulative density function of a $N(0,1)$. Likewise, if $s_t = 1$, then let $s_{t+1} = 0$ whenever $\epsilon_{t+1} > \underline{\epsilon}_0$, such that

$$\psi_{11} = 1 - \Phi(\underline{\epsilon}_1).$$

To find the constants $\gamma(0)$ and $\gamma(1)$, we need to evaluate

$$\begin{aligned} & \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\underline{\epsilon}} \exp\left[-\frac{1}{2\sigma^2}(\epsilon + \sigma^2/\psi)^2\right] d\epsilon \\ &= \mathbb{P}\left\{\frac{\epsilon + \sigma^2/\psi}{\sigma} \le \frac{\underline{\epsilon} + \sigma^2/\psi}{\sigma}\right\} = \Phi\left(\frac{\underline{\epsilon} + \sigma^2/\psi}{\sigma}\right). \end{aligned}$$

Thus, condition (A.4) turns into

$$\begin{aligned} \frac{1}{r\pi_*} &= \left[\gamma(0)\Phi(\underline{\epsilon}_0)\Phi\left(\frac{\underline{\epsilon}_0 + \sigma^2/\psi}{\sigma}\right) + \gamma(1)(1-\Phi(\underline{\epsilon}_0))\left(1 - \Phi\left(\frac{\underline{\epsilon}_0 + \sigma^2/\psi}{\sigma}\right)\right)\right] \exp[\sigma^2/(2\psi^2)] \\ \gamma^\psi(1) &= \left[\gamma(1)(1-\Phi(\underline{\epsilon}_1))\left(1 - \Phi\left(\frac{\underline{\epsilon}_1 + \sigma^2/\psi}{\sigma}\right)\right) + \gamma(0)\Phi(\underline{\epsilon}_1)\Phi\left(\frac{\underline{\epsilon}_1 + \sigma^2/\psi}{\sigma}\right)\right] \exp[\sigma^2/(2\psi^2)]. \end{aligned}$$

This system of two equations can be solved for $\gamma(0)$ and $\gamma(1)$ as a function of the thresholds $\underline{\epsilon}_0$ and $\underline{\epsilon}_1$. Then (A.6) is a stable solution of (A.1) provided that

$$[\gamma(0)]^\psi \le \frac{1}{r\pi_*} \quad \text{and} \quad [\gamma(1)]^\psi \ge \frac{1}{r\pi_*}.$$

Benhabib, Schmitt-Grohé, and Uribe (2001a) Dynamics. BSGU constructed equilibria in which the economy transitioned from the targeted-inflation equilibrium to the deflation equilibrium. Consider the following law of motion for inflation

$$\pi_t^{(BGSU)} = \pi_*\gamma_* \exp[-\epsilon_t/\psi] \exp\big[-\psi^{t-t_0}\big]. \tag{A.7}$$

Here, γ_* was defined in (A.5) and $-t_0$ can be viewed as the initialization period for the inflation process. We need to verify that $\pi_t^{(BGSU)}$ satisfies (A.1). From the derivations that lead to (A.4) we deduce that

$$\gamma_* \mathbb{E}_{t+1}\big[\exp[-\epsilon_{t+1}/\psi]\big] = \gamma_*^\psi.$$

Since

$$\exp\big[-\psi^{t+1-t_0}\big] = \big(\exp\big[-\psi^{t-t_0}\big]\big)^\psi,$$

we deduce that the law of motion for $\pi_t^{(BGSU)}$ in (A.7) satisfies the relationship

$$\mathbb{E}_t[\pi_{t+1}] = \pi_* \left(\frac{\pi_t}{\pi_*}\right)^\psi \exp[\epsilon_t].$$

Moreover, since $\psi > 1$ the term $\exp\big[-\psi^{t-t_0}\psi\big] \longrightarrow 0$ as $t \longrightarrow \infty$. Thus, the economy will move away from the targeted-inflation equilibrium and at some suitably defined t_* reach the deflation equilibrium and remain there permanently. Overall the inflation dynamics take the form

$$\pi_t = \pi_* \begin{cases} \gamma_* \exp[-\epsilon_t/\psi] \exp\big[-\psi^{t-t_0}\big] & \text{if } t \le t_* \\ \gamma_D \exp[-\epsilon_t/\psi] & \text{otherwise} \end{cases}, \tag{A.8}$$

where γ_* and γ_D were defined in (A.5).

Alternative Deflation Equilibria. Around the deflation steady state the system is locally indeterminate. This suggests that we can construct alternative solutions to (A.1). Consider the following conjecture for inflation

$$\pi_t = \pi_* \gamma \min\big\{ \exp[-c/\psi], \exp[-\epsilon/\psi]\big\}, \tag{A.9}$$

where c is a cutoff value. The intuition for this solution is the following. Large positive shocks ϵ that could push the nominal interest rate above one, are off-set by downward movements in inflation. Negative shocks do not need to be off-set, because they push the desired gross interest rate below one and the max operator in the policy rule keeps the interest rate at one. Formally, we can compute the expected value of inflation as follows:

$$\begin{aligned} \mathbb{E}_t[\pi_{t+1}] &= \pi_*\gamma \Bigg[\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{c} \exp[-c/\psi] \exp\left[-\frac{1}{2\sigma^2}\epsilon^2\right] d\epsilon \\ & \quad \frac{1}{\sqrt{2\pi\sigma^2}} \int_{c}^{\infty} \exp[-\epsilon/\psi] \exp\left[-\frac{1}{2\sigma^2}\epsilon^2\right] d\epsilon \\ &= \pi_*\gamma \Bigg[\exp[-c/\psi]\Phi(c/\sigma) + \exp\left[\frac{\sigma^2}{2\psi^2}\right] \int_c^\infty \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(\epsilon + \sigma^2/\psi)^2\right] d\epsilon \Bigg] \\ &= \pi_*\gamma \Bigg[\exp[-c/\psi]\Phi(c/\sigma) + \exp\left[\frac{\sigma^2}{2\psi^2}\right] \left(1 - \Phi\left(\frac{c}{\sigma} + \frac{\sigma}{\psi}\right)\right) \Bigg] \end{aligned} \tag{A.10}$$

Here $\Phi(\cdot)$ denotes the cdf of a standard Normal random variable. Now define

$$f(c,\psi,\sigma) = \left[\exp[-c/\psi]\Phi(c/\sigma) + \exp\left[\frac{\sigma^2}{2\psi^2}\right] \left(1 - \Phi\left(\frac{c}{\sigma} + \frac{\sigma}{\psi}\right)\right) \right].$$

Then another solution for which interest rates stay at the ZLB is given by

$$\bar{\gamma} = \frac{1}{r_*\pi_* f(c,\psi,\sigma)}$$

It can be verified that for c small enough the condition

$$\frac{1}{r_*\pi_*} \ge \bar{\gamma}^{\psi} \min\Big\{ \exp[-c+\epsilon], 1 \Big\}$$

is satisfied.

B Model Solution

The equilibrium conditions (in terms of detrended variables, i.e., $c_t = C_t/A_t$ and $y_t = Y_t/A_t$) take the form

$$1 = \beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\tau} \frac{1}{\gamma z_{t+1}} \frac{R_t}{\pi_{t+1}} \right] \quad \text{(A.11)}$$

$$\begin{aligned} 1 &= \frac{1}{\nu}(1 - c_t^\tau) + \phi(\pi_t - \bar{\pi}) \left[\left(1 - \frac{1}{2\nu} \right) \pi_t + \frac{\bar{\pi}}{2\nu} \right] \quad \text{(A.12)} \\ &\quad -\phi\beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\tau} \frac{y_{t+1}}{y_t} (\pi_{t+1} - \bar{\pi}) \pi_{t+1} \right] \end{aligned}$$

$$c_t = \left[\frac{1}{g_t} - \frac{\phi}{2} (\pi_t - \bar{\pi})^2 \right] y_t \quad \text{(A.13)}$$

$$R_t = \max \left\{ 1, \left[r\pi_* \left(\frac{\pi_t}{\pi_*} \right)^{\psi_1} \left(\frac{y_t}{y_{t-1}} z_t \right)^{\psi_2} \right]^{1-\rho_R} R_{t-1}^{\rho_R} e^{\sigma_R \epsilon_{R,t}} \right\}. \quad \text{(A.14)}$$

B.1 Approximation Near the Targeted-Inflation Steady State

Steady State. Steady state inflation equals π_*. Let $\lambda = \nu(1-\beta)$, then

$$\begin{aligned} r &= \gamma/\beta \\ R_* &= r\pi_* \\ c_* &= \left[1 - v - \frac{\phi}{2}(1 - 2\lambda) \left(\pi_* - \frac{1-\lambda}{1-2\lambda}\bar{\pi} \right)^2 + \frac{\phi}{2} \frac{\lambda^2}{1-2\lambda} \bar{\pi}^2 \right]^{1/\tau} \\ y_* &= \frac{c_*}{\left[\frac{1}{g_*} - \frac{\phi}{2}(\pi_* - \bar{\pi})^2 \right]}. \end{aligned}$$

Log-linearization. We omit the hats from variables that capture deviations from the targeted-inflation steady state. The linearized consumption Euler equation (A.11) is

$$c_t = \mathbb{E}_t[c_{t+1}] - \frac{1}{\tau}(R_t - \mathbb{E}_t[\pi_{t+1} + z_{t+1}]).$$

The price setting equation (A.12) takes the form

$$\begin{aligned} 0 &= -\frac{\tau c_*^\tau}{\nu} c_t + \phi\pi_* \left[\left(1 - \frac{1}{2\nu} \right) \pi_* + \frac{\bar{\pi}}{2\nu} \right] \pi_t + \phi\pi_*(\pi_* - \bar{\pi}) \left(1 - \frac{1}{2\nu} \right) \pi_t \\ &\quad -\phi\beta\pi_*(\pi_* - \bar{\pi}) \Big(\tau c_t - y_t - \mathbb{E}_t[\tau c_{t+1} - y_{t+1}] + \mathbb{E}[\pi_{t+1}] \Big) - \phi\beta\pi_*^2 \mathbb{E}_t[\pi_{t+1}]. \end{aligned}$$

Log-linearizing the aggregate resource constraint (A.12) yields

$$c_t = y_t - \frac{1/g_*}{1/g_* - \phi(\pi_* - \bar{\pi})^2} g_t - \frac{\phi \pi_* (\pi_* - \bar{\pi})}{1/g_* - \phi(\pi_* - \bar{\pi})^2} \pi_t$$

Finally, the monetary policy rule becomes

$$R_t = \max \left\{ -\ln(r\pi_*),\ (1-\rho_R)\psi_1 \pi_t + (1-\rho_R)\psi_2 (y_t - y_{t-1} + z_t) + \rho R_{t-1} + \sigma_R \epsilon_{R,t} \right\}.$$

Approximate Piecewise-Linear Solution in Special Case. To simplify the exposition we impose the following restrictions on the DSGE model parameters: $\tau = 1$, $\gamma = 1$, $\bar{\pi} = \pi_*$, $\psi_1 = \psi$, $\psi_2 = 0$, $\rho_R = 0$, $\rho_z = 0$, and $\rho_g = 0$. We obtain the system

$$\begin{aligned} R_t &= \max \left\{ -\ln(r\pi_*),\ \psi \pi_t + \sigma_R \epsilon_{R,t} \right\} \\ c_t &= \mathbb{E}_t[c_{t+1}] - (R_t - \mathbb{E}_t[\pi_{t+1}]) \\ \pi_t &= \beta \mathbb{E}_t[\pi_{t+1}] + \kappa c_t. \end{aligned} \tag{A.15}$$

It is well known that if the shocks are small enough such that the ZLB is non-binding, the linearized system has a unique stable solution for $\psi > 1$. Since the exogenous shocks are *iid* and the simplified system has no endogenous propagation mechanism, consumption, output, inflation, and interest rates will also be *iid* and can be expressed as a function of $\epsilon_{R,t}$. In turn, the conditional expectations of inflation and consumption equal their unconditional means which we denote by μ_π and μ_c, respectively.

The Euler equation in (A.15) simplifies to the static relationship

$$c_t = -R_t + \mu_c + \mu_\pi. \tag{A.16}$$

Similarly, the Phillips curve in (A.15) becomes

$$\pi_t = \kappa c_t + \beta \mu_\pi. \tag{A.17}$$

Combining (A.16) and (A.17) yields

$$\pi_t = -\kappa R_t + (\kappa + \beta)\mu_\pi + \kappa \mu_c. \tag{A.18}$$

We now can use (A.18) to eliminate inflation from the monetary policy rule:

$$R_t = \max \left\{ -\ln(r\pi_*),\ -\kappa\psi R_t + (\kappa + \beta)\psi \mu_\pi + \kappa \psi \mu_c + \sigma_R \epsilon_{R,t} \right\} \tag{A.19}$$

Define

$$R_t^{(1)} = -\ln(r\pi_*) \quad \text{and} \quad R_t^{(2)} = \frac{1}{1+\kappa\psi}\Big[(\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t}\Big].$$

Let $\bar{\epsilon}_{R,t}$ be the value of the monetary policy shock for which $R_t = -\ln(r\pi_*)$ and the two terms in the max operator of (A.19) are equal

$$\sigma_R\bar{\epsilon}_{R,t} = -(1+\kappa\psi)\ln(r\pi_*) - (\kappa+\beta)\psi\mu_\pi - \kappa\psi\mu_c.$$

To complete the derivation of the equilibrium interest rate it is useful to distinguish the following two cases. Case (i): suppose that $\epsilon_{R,t} < \bar{\epsilon}_{R,t}$. We will verify that $R_t = R_t^{(1)}$ is consistent with (A.19). If the monetary policy shock is less than the threshold value, then

$$(\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\bar{\epsilon}_{R,t} < -(1+\kappa\psi)\ln(r\pi_*).$$

Thus,

$$-\kappa\psi R_t^{(1)} + (\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t} < -\kappa\psi R_t^{(1)} - (1+\kappa\psi)\ln(r\pi_*) = -\ln(r\pi_*),$$

which confirms that (A.19) is satisfied.

Case (ii): suppose that $\epsilon_{R,t} > \bar{\epsilon}_{R,t}$. We will verify that $R_t = R_t^{(2)}$ is consistent with (A.19). If the monetary policy shock is greater than the threshold value, then

$$(\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\bar{\epsilon}_{R,t} > -(1+\kappa\psi)\ln(r\pi_*).$$

In turn,

$$\begin{aligned}
&-\kappa\psi R_t^{(2)} + (\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t} \\
&= -\frac{\kappa\psi}{1+\kappa\psi}\Big[(\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t}\Big] + (\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t} \\
&= \frac{1}{1+\kappa\psi}\Big[(\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t}\Big] \\
&> -\ln(r\pi_*),
\end{aligned}$$

which confirms that (A.19) is satisfied.

We can now deduce that

$$R_t(\epsilon_{R,t}) = \max\left\{-\ln(r\pi_*),\ \frac{1}{1+\kappa\psi}\Big[\psi(\kappa+\beta)\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t}\Big]\right\}. \tag{A.20}$$

Combining (A.16) and (A.20) yields equilibrium consumption

$$c_t(\epsilon_{R,t}) = \begin{cases} \frac{1}{1+\kappa\psi}\Big[(1-\psi\beta)\mu_\pi + \mu_c - \sigma_R\epsilon_{R,t}\Big] & \text{if } R_t \geq -\ln(r\pi_*) \\ \ln(r\pi_*) + \mu_c + \mu_\pi & \text{otherwise} \end{cases}. \tag{A.21}$$

Likewise, combining (A.17) and (A.20) delivers equilibrium inflation

$$\pi_t(\epsilon_{R,t}) = \begin{cases} \frac{1}{1+\kappa\psi}\Big[(\kappa+\beta)\mu_\pi + \kappa\mu_c - \kappa\sigma_R\epsilon_{R,t}\Big] & \text{if } R_t \geq -\ln(r\pi_*) \\ \kappa\ln(r\pi_*) + (\kappa+\beta)\mu_\pi + \kappa\mu_c & \text{otherwise} \end{cases}. \tag{A.22}$$

If $X \sim N(\mu, \sigma^2)$ and C is a truncation constant, then

$$\mathbb{E}[X|X \geq C] = \mu + \frac{\sigma\phi_N(\alpha)}{1-\Phi_N(\alpha)},$$

where $\alpha = (C-\mu)/\sigma$, $\phi_N(x)$ and $\Phi_N(\alpha)$ are the probability density function (pdf) and the cumulative density function (cdf) of a $N(0,1)$. Define the cutoff value

$$C = -(1+\kappa\psi)\ln(r\pi_*) - (\kappa+\beta)\psi\mu_\pi - \kappa\psi\mu_c. \tag{A.23}$$

Using the definition of a cdf and the formula for the mean of a truncated Normal random variable, we obtain that

$$\begin{aligned} \mathbb{P}[\epsilon_{R,t} \geq C/\sigma_R] &= 1 - \Phi_N(C_y/\sigma_R) \\ \mathbb{E}[\epsilon_{R,t}\,|\,\epsilon_{R,t} \geq C/\sigma_R] &= \frac{\sigma_R\phi_N(C/\sigma_R)}{1-\Phi_N(C/\sigma_R)}. \end{aligned}$$

Thus,

$$\begin{aligned} \mu_c &= \frac{1-\Phi_N(C_y/\sigma_R)}{1+\kappa\psi}\Big[(1-\psi\beta)\mu_\pi + \mu_c\Big] - \frac{\sigma_R\phi_N(C_y/\sigma_R)}{(1+\kappa\psi)(1-\Phi_N(C_y/\sigma_R))} && \text{(A.24)} \\ &\quad + \Phi_N(C_y/\sigma_R)\Big[\ln(r\pi_*) + \mu_c + \mu_\pi\Big] \\ \mu_\pi &= \frac{1-\Phi_N(C_y/\sigma_R)}{1+\kappa\psi}\Big[(\kappa+\beta)\mu_\pi + \kappa\mu_c\Big] - \frac{\kappa\sigma_R\phi_N(C_y/\sigma_R)}{(1+\kappa\psi)(1-\Phi_N(C_y/\sigma_R))} && \text{(A.25)} \\ &\quad + \Phi_N(C_y/\sigma_R)\Big[\kappa\ln(r\pi_*) + (\kappa+\beta)\mu_\pi + \kappa\mu_c\Big] \end{aligned}$$

The constants C, μ_c and μ_π can be obtained by solving the system of nonlinear equations comprised of (A.23) to (A.25).

B.2 Approximation Near the Deflation Steady State

Steady State. As before, let $\lambda = \nu(1-\beta)$. The steady state nominal interest rate is $R_D = 1$ and provided that $\beta/(\gamma\pi_*) < 1$ and $\psi_1 > 1$:

$$\begin{aligned} r &= \gamma/\beta \\ \pi_D &= \beta/\gamma \\ c_D &= \left[1 - v - \frac{\phi}{2}(1-2\lambda)\left(\pi_D - \frac{1-\lambda}{1-2\lambda}\bar{\pi}\right)^2 + \frac{\phi}{2}\frac{\lambda^2}{1-2\lambda}\bar{\pi}^2\right]^{1/\tau} \\ y_D &= \frac{c_D}{\left[\frac{1}{g_*} - \frac{\phi}{2}(\pi_D - \bar{\pi})^2\right]}. \end{aligned}$$

Log-linearization. We omit the tildes from variables that capture deviations from the deflation steady state. The linearized consumption Euler equation (A.11) is

$$c_t = \mathbb{E}_t[c_{t+1}] - \frac{1}{\tau}(R_t - \mathbb{E}_t[\pi_{t+1} + z_{t+1}]).$$

The price setting equation (A.12) takes the form

$$\begin{aligned} 0 \;=\; & -\frac{\tau c_D^\tau}{\nu}c_t + \phi\beta\left[\left(1 - \frac{1}{2\nu}\right)\beta + \frac{\bar{\pi}}{2\nu}\right]\pi_t + \phi\beta(\beta - \bar{\pi})\left(1 - \frac{1}{2\nu}\right)\pi_t \\ & -\phi\beta^2(\beta - \bar{\pi})\Big(\tau c_t - y_t - \mathbb{E}_t[\tau c_{t+1} - y_{t+1}] + \mathbb{E}[\pi_{t+1}]\Big) - \phi\beta^3\mathbb{E}_t[\pi_{t+1}]. \end{aligned}$$

Log-linearizing the aggregate resource constraint (A.12) yields

$$c_t = y_t - \frac{1/g_*}{1/g_* - \phi(\beta - \bar{\pi})^2}g_t - \frac{\phi\beta(\beta - \bar{\pi})}{1/g_* - \phi(\beta - \bar{\pi})^2}\pi_t$$

Finally, the monetary policy rule becomes

$$\begin{aligned} R_t \;=\; \max\Big\{ & 0,\; -(1-\rho_R)\ln(r\pi_*) - (1-\rho_R)\psi_1\ln(\pi_*/\beta) \\ & +(1-\rho_R)\psi_1\pi_t + (1-\rho_R)\psi_2(y_t - y_{t-1} + z_t) + \rho R_{t-1} + \sigma_R\epsilon_{R,t}\Big\}. \end{aligned}$$

Approximate Piecewise-Linear Solution in Special Case. As for the approximate analysis of the targeted-inflation equilibrium, we impose the following restrictions on the DSGE model parameters: $\tau = 1$, $\gamma = 1$, $\bar{\pi} = \pi_*$, $\psi_1 = \psi$, $\psi_2 = 0$, $\rho_R = 0$, $\rho_z = 0$, and

$\rho_g = 0$. In the deflation equilibrium the steady state inflation rate is $\pi_D = \beta$. To ease the expositions we assume that the terms $|\pi_D - \bar{\pi}|$ that appear in the log-linearized equations above are negligible. Denote percentage deviations of a variable x_t from its deflation steady state by $\tilde{x}_t = \ln(x_t/x_D)$. If we let $\kappa_D = c_D/(\nu\phi\beta^2)$ and using the steady state relationship $r = 1/\beta$

$$\begin{aligned} \tilde{R}_t &= \max\left\{0,\ -(\psi-1)\ln(r\pi_*) + \psi\tilde{\pi}_t + \sigma_R\epsilon_{R,t}\right\} \\ \tilde{c}_t &= \mathbb{E}_t[\tilde{c}_{t+1}] - (\tilde{R}_t - \mathbb{E}_t[\tilde{\pi}_{t+1}]) \\ \tilde{\pi}_t &= \beta\mathbb{E}_t[\tilde{\pi}_{t+1}] + \kappa_D\tilde{c}_t. \end{aligned} \tag{A.26}$$

Provided that $\psi > 1$, the ZLB is binding with high probability if the shock standard deviation σ_R is small. In this case $\tilde{R}_t = 0$. An equilibrium in which all variables are *iid* can be obtained by adjusting the constants in (A.20) to (A.22):

$$\begin{aligned} \tilde{R}_t(\epsilon_{R,t}) &= \max\left\{0,\ \frac{1}{1+\kappa\psi}\left[\psi(\kappa+\beta)\mu_\pi^D + \kappa\psi\mu_c^D - (\psi-1)\ln(r\pi_*) + \sigma_R\epsilon_{R,t}\right]\right\} \\ \tilde{c}_t(\epsilon_{R,t}) &= \begin{cases} \frac{1}{1+\kappa\psi}\left[(1-\psi\beta)\mu_\pi^D + \mu_c^D + (\psi-1)\ln(r\pi_*) - \sigma_R\epsilon_{R,t}\right] & \text{if } \tilde{R}_t \geq 0 \\ \mu_c^D + \mu_\pi^D & \text{otherwise} \end{cases} \\ \tilde{\pi}_t(\epsilon_{R,t}) &= \begin{cases} \frac{1}{1+\kappa\psi}\left[(\kappa+\beta)\mu_\pi^D + \kappa\mu_c^D + \kappa(\psi-1)\ln(r\pi_*) - \kappa\sigma_R\epsilon_{R,t}\right] & \text{if } \tilde{R}_t \geq 0 \\ (\kappa+\beta)\mu_\pi^D + \kappa\mu_c^D & \text{otherwise} \end{cases}. \end{aligned} \tag{A.27}$$

In this simple model, the decision rules have a kink at the point in the state space where the two terms in the max operator of the interest rate equation are equal to each other. In the targeted-inflation equilibrium this point in the state space is given by

$$\bar{\epsilon}_R^* = \frac{1}{\sigma_R}\left[-(1+\kappa\psi)\ln(r\pi_*) - (\kappa+\beta)\psi\mu_\pi^* - \kappa\psi\mu_c^*\right],$$

whereas in the deflation equilibrium it is

$$\bar{\epsilon}_R^D = \frac{1}{\sigma_R}\left[(\psi-1)\ln(r\pi_*) - (\kappa+\beta)\psi\mu_\pi^D - \kappa\psi\mu_c^D\right],$$

Once $\epsilon_{R,t}$ falls below the threshold value $\bar{\epsilon}_R^*$ or $\bar{\epsilon}_R^D$, its marginal effect on the endogenous variables is zero. To the extent that $\bar{\epsilon}_R^D > 0 > \bar{\epsilon}_R^*$, it takes a positive shock in the deflation equilibrium to move away from the ZLB, whereas it takes a large negative monetary shock in the targeted-inflation equilibrium to hit the ZLB.

C Computational Details

C.1 Model Solution Algorithm

Algorithm 1 (Solution Algorithm) *1. Start with a guess for Θ. For the targeted-inflation equilibrium, this guess is obtained from a linear approximation around the inflation target. For the deflation equilibrium, it is obtained by assuming constant decision rules for inflation and $\mathcal{E}$ at the deflation steady state. For the sunspot equilibrium it is obtained by letting the $s_t = 1$ decision rules come from the targeted-inflation equilibrium and the $s_t = 0$ decision rules come from the deflation equilibrium.*

2. *Given this guess, simulate the model for a large number of periods.*

3. *Given the simulated path, obtain the grid for the state variables over which the approximation needs to be accurate. Label these grid points as $\{\mathcal{S}_1, ..., \mathcal{S}_M\}$. For a fourth-order approximation, we use $M = 130$. For the targeted-inflation equilibrium, 79 of these grid points come from the ergodic distribution, obtained using a cluster-grid algorithm as in Judd, Maliar, and Maliar (2010). The remaining 51 come from the filtered exogenous state variables from 2000Q1 to 2012Q3. For the deflation equilibrium, we use a time-separated grid algorithm to deliver* 130 *points, which suits the behavior of this equilibrium better, since there are many periods where the economy is on the "edge" of the ergodic distribution at the ZLB. For the sunspot equilibrium, we use the same time-separated grid algorithm to deliver 156 points each for $s_t = 1$ and $s_t = 0$ and 312 points come from filtered states, using multiple particles per period from the particle filter, and over-sampling the period 2009Q2-2011Q2.*

4. *Solve for the Θ by minimizing the sum of squared residuals obtained following the steps below, using a variant of a Newton algorithm.*

   (a) *For a generic grid point $\mathcal{S}_i$, and the current value for Θ, compute $f^1_\pi(\mathcal{S}_i;\Theta)$, $f^2_\pi(\mathcal{S}_i;\Theta)$, $f^1_\mathcal{E}(\mathcal{S}_i;\Theta)$ and $f^2_\mathcal{E}(\mathcal{S}_i;\Theta)$.*

   (b) *Assume $\zeta_i \equiv I\{R(\mathcal{S}_i, \Theta) > 1\} = 1$ and compute π_i, $\mathcal{E}_i$, as well as y_i and c_i using (23) and (24), substituting in (25).*

(c) *If R_i that follows from (25) using π_i and y_i obtained in (b) is greater than unity, then ζ_i is indeed equal to one. Otherwise, set $\zeta_i = 0$ (and thus $R_i = 1$) and recompute all other objects.*

(d) *The final step is to compute the residual functions. There are four residuals, corresponding to the four functions being approximated. For a given set of state variables $\mathcal{S}_i$, only two of them will be relevant since we either need the constrained decision rules or the unconstrained ones. The residual functions will be given by*

$$\mathcal{R}^1(\mathcal{S}_i) = \mathcal{E}_i - \left[\int \int \int \frac{c(\mathcal{S}')^{-\tau}}{\gamma z' \pi(\mathcal{S}')} dF\left(z'\right) dF\left(g'\right) dF\left(\epsilon_R'\right) \right] \tag{A.28}$$

$$\mathcal{R}^2(\mathcal{S}_i) = f\left(c_i, \pi_i, y_i\right) - \phi\beta \int \int \int c(\mathcal{S}')^{-\tau} y(\mathcal{S}') \left[\pi(\mathcal{S}') - \bar{\pi}\right] \pi(\mathcal{S}') dF\left(z'\right) dF\left(g'\right) dF\left(\epsilon_R'\right) \tag{A.29}$$

Note that this step involves computing $\pi(\mathcal{S}')$, $y(\mathcal{S}')$, $c(\mathcal{S}')$ and $R(\mathcal{S}')$ which is done following steps (a)-(c) above for each value of $\mathcal{S}'$. We use a non-product monomial integration rule to evaluate these integrals.

(e) *The objective function to be minimized is the sum of squared residuals obtained in (d).*

5. *Repeat steps 2-4 sufficient number of times so that the ergodic distribution remains unchanged from one iteration to the next. For the targeted-inflation equilibrium and the sunspot equilibrium, we also iterate between solution and filtering to make sure the filtered states used in the solution grid remain unchanged.*

We start our solution from a second-order approximation and move to a third- and fourth-order approximation by using the previous solution. We use analytical derivatives of the objection function which speeds up the solution by two orders of magnitude. As a measure of accuracy, we compute the approximation errors from A.28 and A.29, converted to consumption units. For the targeted-inflation equilibrium, these are in the order of 10^{-6}. For the deflation and sunspot equilibria they are higher at 10^{-4} and 10^{-5}, respectively, but still very reasonable given the complexity of the model.

Figure A-1: Solution Grid for the Targeted-Inflation Equilibrium

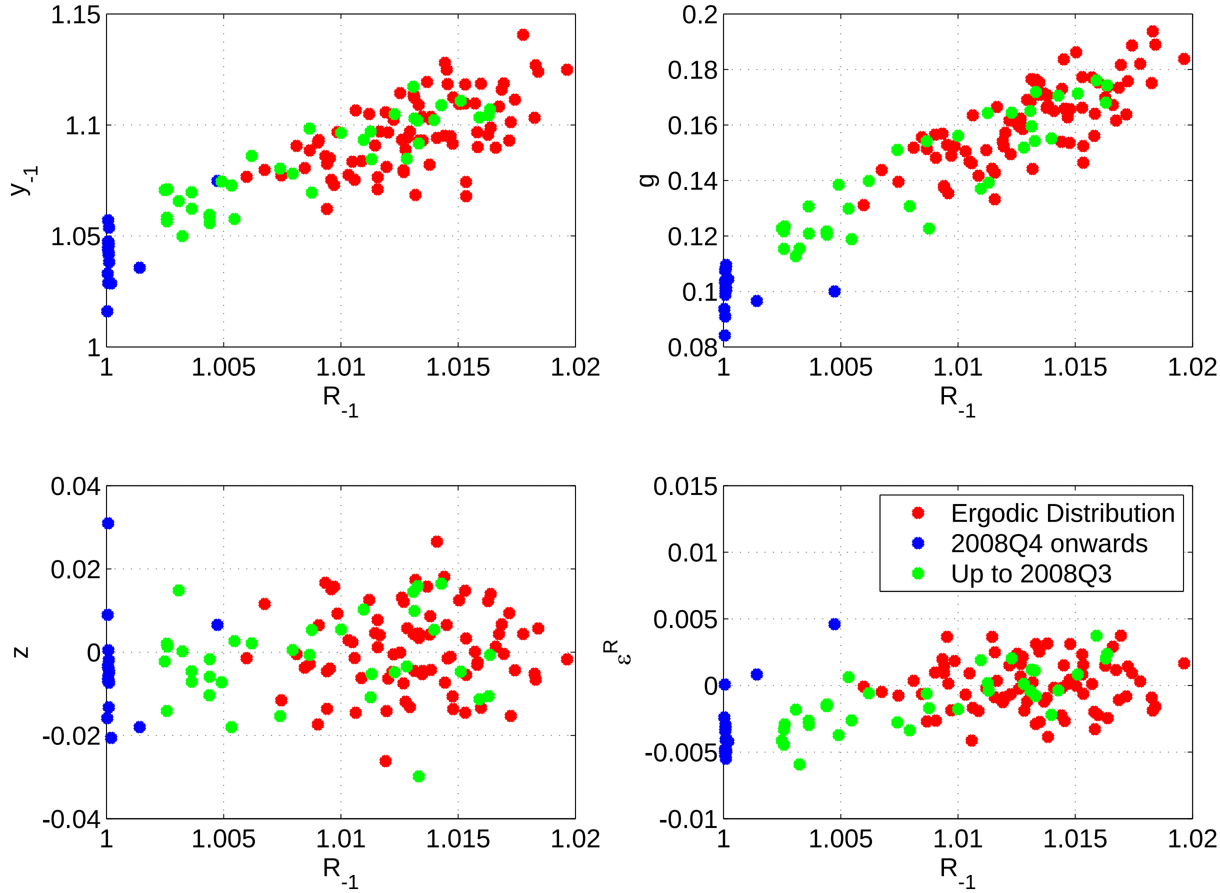

Figure A-1 shows the solution grid for the targeted-inflation equilibrium. For each panel we have R_{t-1} on the x axis and the other state variables on the y axis. The red dots are the grid points that represent the ergodic distribution, the green points are the filtered states from 2000:Q1 to 2008:Q3 and the blue points are the filtered state for the period after 2008:Q3. It is evident that the filtered states lie in the tails of the ergodic distribution of the targeted-inflation equilibrium, which assigns negligible probability to zero interest rates and the exogenous states that push interest rates toward the ZLB. By adding these filtered states to the grid points, we ensure that our approximation will be accurate in these low-probability regions.

C.2 Details of Policy Experiments

Algorithm 2 (Effect of Combined Fiscal and Monetary Policy Intervention) *For $j = 1$ to $j = n_{sim}$ repeat the following steps:*

1. *Initialize the simulation by setting $(R_0^{(j)}, y_0^{(j)}, z_0^{(j)}, g_0^{(j)})$ equal to the mean estimate obtained with the particle filter.*

2. *Generate baseline trajectories based on the innovation sequence $\{\epsilon_t^{(j)}\}_{t=1}^H$ by letting $[\epsilon_{z,t}^{(j)}, \epsilon_{g,t}^{(j)}]' \sim N(0, I)$ and setting $\epsilon_{R,t} = 0$.*

3. *Generate the innovation sequence for the counterfactual trajectories according to*

$$\begin{aligned} \epsilon_{g,1}^{I(j)} &= \delta^{ARRA} + \epsilon_{g,1}^{(j)}; \quad \epsilon_{g,t}^{I(j)} = \epsilon_{g,t}^{(j)} \quad \textit{for} \quad t = 2, \dots, H; \\ \epsilon_{z,t}^{I(j)} &= \epsilon_{z,t}^{(j)} \quad \textit{for} \quad t = 1, \dots, H; \\ \epsilon_{R,t}^{I(j)} &= \epsilon_{R,t}^{(j)} = 0 \quad \textit{for} \quad t = 9, \dots, H; \end{aligned}$$

   In periods $t = 1, \dots, 8$, conditional on $\{\epsilon_{g,t}^{I(j)}, \epsilon_{z,t}^{I(j)}\}_{t=1}^4$, determine $\epsilon_{R,t}^{I(j)}$ by solving for the smallest $\tilde{\epsilon}_{R,t}$ such that it is less than $2\sigma_R$ in absolute value, that yields either

$$R_t^{I(j)}(\epsilon_{R,t}^{I(j)} = \tilde{\epsilon}_{R,t}) = 1 \quad \textit{or} \quad 400 \ln \left(R_t^{I(j)}(\epsilon_{R,t}^{I(j)} = 0) - R_t^{I(j)}(\epsilon_{R,t}^{I(j)} = \tilde{\epsilon}_{R,t}) \right) = 1.$$

4. *Conditional on $(R_0^{(j)}, y_0^{(j)}, z_0^{(j)}, g_0^{(j)})$ compute $\{R_t^{(j)}, y_t^{(j)}, \pi_t^{(j)}\}_{t=1}^H$ and $\{R_t^{I(j)}, y_t^{I(j)}, \pi_t^{I(j)}\}_{t=1}^H$ based on $\{\epsilon_t^{(j)}\}$ and $\{\epsilon_t^{I(j)}\}$, respectively, and let*

$$IRF^{(j)}(x_t | \epsilon_{g,1}, \epsilon_{R,1:8}) = (\ln x_t^{I(j)} - \ln x_t^{(j)}). \tag{A.30}$$

Compute medians and percentile bands based on $IRF^{(j)}(x_t | \epsilon_{g,1}, \epsilon_{R,1:8})$, $j = 1, \dots, n_{sim}$. □

When we consider only a fiscal policy, we set $\epsilon_{R,t}^{I(j)} = 0$ for $t = 1, ..., 8$ as well.

D Estimation of 2nd-Order Approximated DSGE Model

Table A-1 summarizes the prior and posterior distribution from the Bayesian estimation of the 2nd-order approximated version of the DSGE model. The estimation sample is 1984:Q1 to 2007:Q4. The parameter ϕ that is used in the main text is related to the parameter κ (Phillips curve slope of a linearized version of the DSGE model) according to $\phi = \frac{\tau(1-\nu)}{(\nu\pi^2\kappa)}$. The parameters r^*, π^*, and γ are fixed at the sample means of the ex-post real rate, the inflation rate, and output growth. We assume that $\bar{\pi} = 1$, meaning that any price change is costly.

Table A-1: Posterior Estimates for DSGE Model Parameters

		Prior		Posterior	
Parameter	Density	Para 1	Para2	Mean	90% Interval
τ	Gamma	2.00	0.25	1.50	[1.14, 1.89]
κ	Gamma	0.30	0.10	0.17	[0.05, 0.30]
ψ_1	Gamma	1.50	0.10	1.36	[1.27, 1.43]
ρ_r	Beta	0.50	0.20	0.64	[0.55, 0.72]
ρ_g	Beta	0.80	0.10	0.86	[0.82, 0.91]
ρ_z	Beta	0.20	0.10	0.11	[0.03, 0.24]
$100\sigma_r$	Inv Gamma	0.30	4.00	0.21	[0.17, 0.26]
$100\sigma_g$	Inv Gamma	0.40	4.00	0.78	[0.66, 0.93]
$100\sigma_z$	Inv Gamma	0.40	4.00	1.03	[0.83, 1.32]
$400(r^* - 1)$	Fixed	2.78			
$400(\pi^* - 1)$	Fixed	2.52			
$100(\gamma - 1)$	Fixed	0.48			
$\overline{\pi}$	Fixed	1.00			
ψ_2	Fixed	0.80			
ν	Fixed	0.10			
$\frac{1}{g}$	Fixed	0.85			

Notes: Para (1) and Para (2) list the means and the standard deviations for Beta and Gamma; and s and ν for the Inverse Gamma distribution, where $p_{\mathcal{IG}}(\sigma|\nu, s) \propto \sigma^{-\nu-1}e^{-\nu s^2/2\sigma^2}$. The effective prior is truncated at the boundary of the determinacy region. Estimation sample is 1984:Q1 to 2007:Q4. As 90% credible interval we are reporting the 5th and 95th percentile of the posterior distribution.

E Particle Filter

The particle filter is used to extract information about the state variables of the model from data on output growth, inflation, and nominal interest rates over the period 2000:Q1 to 2012:Q3.

E.1 State-Space Representation

Let y_t be the 3×1 vector of observables consisting of output growth, inflation, and nominal interest rates. The vector x_t stacks the continuous state variables which are given by $x_t = [R_t, y_t, y_{t-1}, z_t, g_t, A_t]'$ and $s_t \in \{0, 1\}$ is the Markov-switching process.

$$y_t = \Psi(x_t) + \nu_t \tag{A.31}$$

$$\mathbb{P}\{s_t = 1\} = \begin{cases} (1 - p_{00}) & \text{if } s_{t-1} = 0 \\ p_{11} & \text{if } s_{t-1} = 1 \end{cases} \tag{A.32}$$

$$x_t = F_{s_t}(x_{t-1}, \epsilon_t) \tag{A.33}$$

The first equation is the measurement equation, where $\nu_t \sim N(0, \Sigma_\nu)$ is a vector of measurement errors. The second equation represents law of motion of the Markov-switching process. The third equation corresponds to the law of motion of the continuous state variables. The vector $\epsilon_t \sim N(0, I)$ stacks the innovations $\epsilon_{z,t}$, $\epsilon_{g,t}$, and $\epsilon_{R,t}$. The functions $F_0(\cdot)$ and $F_1(\cdot)$ are generated by the model solution procedure. We subsequently use the densities $p(y_t|x_t)$, $p(s_t|s_{t-1})$, and $p(x_t|x_{t-1}, s_t)$ to summarize the measurement and the state transition equations. The targeted-inflation equilibrium yields a state-space system that is a special case: the discrete state s_t is constant.

E.2 Sequential Importance Sampling Approximation

Let $z_t = [x_t', s_t]'$ and $Y_{t_0:t_1} = \{y_{t_0}, \ldots, y_{t_1}\}$. Particle filtering relies on sequential importance sampling approximations. The distribution $p(z_{t-1}|Y_{1:t-1})$ is approximated by a set of pairs

$\{(z_{t-1}^{(i)}, \pi_{t-1}^{(i)})\}_{i=1}^{N}$ in the sense that

$$\frac{1}{N}\sum_{i=1}^{N} f(z_{t-1}^{(i)})\pi_{t-1}^{(i)} \xrightarrow{a.s.} \mathbb{E}[f(z_{t-1})|Y_{1:t-1}], \tag{A.34}$$

where $z_{t-1}^{(i)}$ is the i'th particle, $\pi_{t-1}^{(i)}$ is its weight, and N is the number of particles. An important step in the filtering algorithm is to draw a new set of particles for period t. In general, these particles are drawn from a distribution with a density that is proportional to $g(z_t|Y_{1:t}, z_{t-1}^{(i)})$, which may depend on the particle value in period $t-1$ as well as the observation y_t in period t. This procedure leads to an importance sampling approximation of the form:

$$\begin{aligned}
\mathbb{E}[f(z_t)|Y_{1:t}] &= \int_{z_t} f(z_t)\frac{p(y_t|z_t)p(z_t|Y_{1:t-1})}{p(y_t|Y_{1:t-1})}dz_t \qquad (A.35)\\
&= \int_{z_{t-1:t}} f(z_t)\frac{p(y_t|z_t)p(z_t|z_{t-1})p(z_{t-1}|Y_{1:t-1})}{p(y_t|Y_{1:t-1})}dz_{t-1:t}\\
&\approx \frac{\frac{1}{N}\sum_{i=1}^{N} f(z_t^{(i)})\frac{p(y_t|z_t^{(i)})p(z_t^{(i)}|z_{t-1}^{(i)})}{g(z_t^{(i)}|Y_{1:t}, z_{t-1}^{(i)})}\pi_{t-1}^{(i)}}{\frac{1}{N}\sum_{j=1}^{N}\frac{p(y_t|z_t^{(j)})p(z_t^{(j)}|z_{t-1}^{(j)})}{g(z_t^{(j)}|Y_{1:T}, z_{t-1}^{(i)})}\pi_{t-1}^{(i)}}\\
&= \frac{1}{N}\sum_{i=1}^{N} f(z_t^{(i)})\left(\frac{\tilde{\pi}_t^{(i)}}{\frac{1}{N}\sum_{j=1}^{N}\tilde{\pi}_t^{(j)}}\right) = \frac{1}{N}\sum_{i=1}^{N} f(z_t^{(i)})\pi_t^{(i)},
\end{aligned}$$

where the un-normalized and normalized probability weights are given by

$$\tilde{\pi}_t^{(i)} = \frac{p(y_t|z_t^{(i)})p(z_t^{(i)}|z_{t-1}^{(i)})}{g(z_t^{(i)}|Y_{1:T}, z_{t-1}^{(i)})}\pi_{t-1}^{(i)} \quad \text{and} \quad \pi_t^{(i)} = \frac{\tilde{\pi}_t^{(i)}}{\sum_{j=1}^{N}\tilde{\pi}_t^{(j)}}, \tag{A.36}$$

respectively. In simple versions of the particle filter $z_t^{(i)}$ is often generated by simulating the model forward, which means that $g(z_t^{(i)}|Y_{1:T}, z_{t-1}^{(i)}) \propto p(z_t^{(i)}|z_{t-1}^{(i)})$ and the formula for the particle weights simplifies considerably. Unfortunately, this approach is quite inefficient in our application and we require a more elaborate density $g(\cdot|\cdot)$ described below that accounts for information in y_t. The resulting extension of the particle filter is known as auxiliary particle filter, e.g. Pitt and Shephard (1999).

E.3 Filtering

Initialization. To generate the initial set of particles $\{(z_0^{(i)}, \pi_0^{(i)}\}_{i=1}^{N}$, for each i simulate the DSGE model for T_0 periods, starting from the targeted-inflation steady state, and set

$\pi_0^{(i)} = 1$.

Sequential Importance Sampling. For $t = 1$ to T:

1. $\{z_{t-1}^{(i)}, \pi_{t-1}^{(i)}\}_{i=1}^N$ is the particle approximation of $p(z_{t-1}|Y_{1:t-1})$. For $i = 1$ to N:
   (a) Draw $z_t^{(i)}$ conditional on $z_{t-1}^{(i)}$ from $g(z_t|Y_{1:t}, z_{t-1}^{(i)})$.
   (b) Compute the unnormalized particle weights $\tilde{\pi}_t^{(i)}$ according to (A.36).
2. Compute the normalized particle weights $\pi_t^{(i)}$ and the effective sample size $ESS_t = N^2/\sum_{i=1}^N(\pi_t^{(i)})^2$.
3. Resample the particles via deterministic resampling (see Kitagawa (1996)). Reset weights to be $\pi_t^{(i)} = 1$ and approximate $p(z_t|Y_{1:t})$ by $\{(z_t^{(i)}, \pi_t^{(i)})\}_{i=1}^n$.

E.4 Tuning of the Filter

In the empirical analysis we set $T_0 = 50$ and $N = 500,000$. We also fix the measurement error standard deviations for output growth, inflation, and interest rates at 0.1, respectively. Since our model has discrete and continuous state variables, we write

$$p(z_t|z_{t-1}) = \begin{cases} p_0(x_t|x_{t-1}, s_t = 0)\mathbb{P}\{s_t = 0|s_{t-1}\} & \text{if } s_t = 0 \\ p_1(x_t|x_{t-1}, s_t = 1)\mathbb{P}\{s_t = 1|s_{t-1}\} & \text{if } s_t = 1 \end{cases}$$

and consider proposal densities of the form

$$q(z_t|z_{t-1}, y_t) = \begin{cases} q_0(x_t|x_{t-1}, y_t, s_t = 0)\lambda(z_{t-1}, y_t) & \text{if } s_t = 0 \\ q_1(x_t|x_{t-1}, y_t, s_t = 1)(1 - \lambda(z_{t-1}, y_t)) & \text{if } s_t = 1 \end{cases},$$

where $\lambda(x_{t-1}, y_t)$ is the probability that $s_t = 0$ under the proposal distribution. We use $q(\cdot)$ instead of $g(\cdot)$ to indicate that the densities are normalized to integrate to one.

We effectively generate draws from the proposal density through forward iteration of the state transition equation. To adapt the proposal density to the observation y_t we draw $\epsilon_t^{(i)} \sim N(\mu^{(i)}, \Sigma^{(i)})$ instead of the model-implied $\epsilon_t \sim N(0, I)$. In slight abuse of notation (ignoring that the dimension of x_t is larger than the dimension of ϵ_t and that its distribution is singular), we can apply the change of variable formula to obtain a representation of the proposal density

$$q(x_t^{(i)}|x_{t-1}^{(i)}) = q_\epsilon(F^{-1}(x_t^{(i)}|x_{t-1}^{(i)})) \left| \frac{\partial F^{-1}(x_t^{(i)}|x_{t-1}^{(i)})}{\partial x_t} \right|$$

Using the same change-of-variable formula, we can represent

$$p(x_t^{(i)}|x_{t-1}^{(i)}) = p_\epsilon(F^{-1}(x_t^{(i)}|x_{t-1}^{(i)})) \left| \frac{\partial F^{-1}(x_t^{(i)}|x_{t-1}^{(i)})}{\partial x_t} \right|$$

By construction, the Jacobian terms cancel and the ratio that is needed to calculate the unnormalized particle weights for period t in (A.36) simplifies to

$$\tilde{\pi}_t^{(i)} = p(y_t|z_t^{(i)}) \frac{\exp\left\{-\frac{1}{2}\epsilon_t^{(i)\prime}\epsilon_t^{(i)}\right\}}{|\Sigma_\epsilon^{(i)}|^{-1/2}\exp\left\{-\frac{1}{2}(\epsilon_t^{(i)} - \mu^{(i)})^\prime[\Sigma^{(i)}]^{-1}(\epsilon_t^{(i)} - \mu^{(i)})\right\}} \pi_{t-1}^{(i)}.$$

The choice of μ and Σ is described below.

Targeted-Inflation Equilibrium. Since the discrete state s_t is irrelevant in this equilibrium, let $z_t = x_t$. We break the sample period into two parts: 2000:Q1 to 2008:Q4 and 2009:Q1 to 2012:Q3. In the second period the economy was at the ZLB and the filter requires a different proposal density.

For the first part of the sample we run the Kalman filter for the log-linearized version of the DSGE model in parallel with the particle filter and set $\mu^{(i)} = \epsilon_{t|t}^{(i)}$ and $\Sigma^{(i)} = P_{t|t}^{(i)}$, which are the mean and variance of ϵ_t conditional on $Y_{1:t}$ and $z_{t-1} = z_{t-1}^{(i)}$. For the second part of the sample the Kalman filtered shocks become very inaccurate because the log-linearized DSGE model misses the ZLB. Instead, we let $z_{t-1|t-1}$ be a particle filter approximation of $\mathbb{E}[z_{t-1}|Y_{1:t-1}]$ and define

$$\bar{\pi}_t(\epsilon_t) = p(y_t|F(z_{t-1|t-1}, \epsilon_t)) \exp\left\{-\frac{1}{2}\epsilon_t^\prime\epsilon_t\right\} |\Sigma_\epsilon|^{1/2}\pi_{t-1}^{(i)}.$$

We use a grid search over ϵ_t to determine a value $\bar{\epsilon}$ that maximizes this objective function and then set $\mu^{(i)} = \bar{\epsilon}$. (Executing the grid search conditional on each $z_{t-1}^{(i)}$, $i = 1, \ldots, N$ turned out to be too time consuming.)

Sunspot Equilibrium. The filter is initialized by simulating the model for $T_0 = 50$ periods conditional on $s_t = 1$. For the period of 2000:Q1 to 2008:Q4 we use the simple gridsearch approach described in the previous paragraph to generate shocks under which we simulate the state-transition equation forward. Starting in 2008:Q4 we use the information from the grid search to construct a mixture-of-normals proposal distribution for $\epsilon_t^{(i)}$. While more time consuming, this mixture proposal improves the accuracy of the particle filter. At each iteration we conduct separate computations for $s_t = 0$ and $s_t = 1$. We then compute the posterior odds of $s_t = 0$ and $s_t = 1$ and select the regime-conditional particles accordingly. For the ex-ante policy analysis we run the filter from 2009:Q1 onwards conditional on a sequence of regimes for the periods from 2009:Q2 to 2011:Q1.

F Calibration of the Policy Experiment

Table A-2 summarizes the award and disbursements of funds for federal contracts, grants, and loans. We translate the numbers in the table into a one-period location shift of the distribution of $\epsilon_{g,t}$. In our model total government spending is a fraction ζ_t of aggregate output, where ζ_t evolves according to an exogenous process:

$$G_t = \zeta_t Y_t; \quad \zeta_t = 1 - \frac{1}{g_t}; \quad \ln(g_t/g_*) = \rho_g \ln(g_{t-1}/g_*) + \sigma_g \epsilon_{g,t}$$

For the subsequent calibration of the fiscal intervention it is convenient to define the percentage deviations of g_t and ζ_t from their respective steady states: $\hat{g}_t = \ln(g_t/g_*)$ and $\hat{\zeta}_t = \ln(\zeta_t/\zeta_*)$. According to the parameterization of the DSGE model in Table 1 $\zeta_* = 0.15$ and $g_* = 1.177$. Thus, government spending is approximately 15% of GDP. We assume that the fiscal expansion approximately shifts $\hat{\zeta}_t$ to $\hat{\zeta}_t^I = \hat{\zeta}_t + \hat{\zeta}_t^{ARRA}$.

We construct $\hat{\zeta}_t^{ARRA}$ as follows. Let G_t^{ARRA} correspond to the additional government spending stipulated by ARRA. Since we focus on received rather than awarded funds, G_t^{ARRA} corresponds to the third column of Table A-2. The size of the fiscal expansion as a fraction of GDP is

$$\zeta_t^{ARRA} = G_t^{ARRA}/Y_t,$$

where Y_t here corresponds to the GDP data reported in the last column of Table A-2. We then divide by ζ_* to convert it into deviations from the steady state level: $\hat{\zeta}_t^{ARRA} = \zeta_t^{ARRA}/\zeta_*$.

Taking a log-linear approximation of the relationship between g_t and ζ_t leads to

$$\hat{g}_t^{ARRA} = 0.177 \cdot G_t^{ARRA}/(\zeta_* Y_t).$$

In Figure A-2 we compare $\hat{g}_t^{ARRA}$ constructed from the data in Table A-2 to $(\hat{g}_t^I - \hat{g}_t)$, where $\delta^{ARRA} = 0.011$.[14] While the actual path of the received funds is not perfectly monotone, the calibrated intervention in the DSGE model roughly matches the actual intervention both in terms of magnitude and decay rate.

Table A-2: ARRA Funds for Contracts, Grant, and Loans

	Awarded	Received	Nom. GDP
2009:3	158	36	3488
2009:4	17	18	3533
2010:1	26	8	3568
2010:2	16	24	3603
2010:3	33	26	3644
2010:4	9	21	3684
2011:1	4	19	3704
2011:2	4	20	3751
2011:3	8	17	3791
2011:4	0	12	3830
2012:1	3	9	3870
2012:2	0	8	3899

Notes: Data is obtained from www.recovery.org.

[14]Recall that $\sigma_g = 0.0078$

Figure A-2: Calibration of Fiscal Policy Intervention