Endogenous Labor Share Cycles: Theory and Evidence*

Jakub Growiec[†] Peter McAdam[‡] Jakub Mućk[†]

July 22, 2014

Abstract

Based on long US time series we document a range of empirical properties of the labor's share of GDP, including its substantial medium-run swings. We explore the extent to which these empirical regularities can be explained by a calibrated micro-founded long-run economic growth model with normalized CES technology and endogenous labor- and capital-augmenting technical change driven by purposeful directed R&D investments. It is found that dynamic macroeconomic trade-offs created by arrivals of both types of new technologies may lead to prolonged swings in the labor share due to oscillatory convergence to the balanced growth path as well as stable limit cycles via Hopf bifurcations. Both predictions are broadly in line with the empirical evidence.

Keywords: Labor income share, endogenous cycles, factor-augmenting endogenous technical change, R&D, CES.

JEL Codes: E25, E32, O33.

*We gratefully acknowledge support from the Polish National Science Center (Narodowe Centrum Nauki) under the grant Opus 3 No. 2012/05/B/HS4/02236. The views expressed are those of the authors alone. We thank Łukasz Drozd, seminar participants at the CEF conference (Oslo, 2014), UECE conference (Lisbon, 2014) and the 3rd NBP Summer Workshop (Warsaw, 2014), for helpful comments and suggestions.

[†]Warsaw School of Economics and Narodowy Bank Polski.

[‡]European Central Bank and University of Surrey.

Non Technical Summary

Looking at historical data for the US, we document that there has been a hump-shaped pattern in the labor share, coupled with marked medium-term volatility and high persistence. We show that most of the variance of the labor share lies *beyond* business-cycle frequencies. Therefore explaining labor-share movements with the aid of business-cycle mechanisms will only take us so far. Second, though necessarily stationary, there is no compelling evidence that labor income shares are mean reverting (even in long-dated samples).

One suggestive way of reconciling these two aspects is to consider that labor income shares may be better characterized as being driven by a long cycle. We pursue this idea through the lens of an endogenous growth model. We consider a non-scale model of endogenous R&D-based growth with (a) two R&D sectors, giving rise to capital- as well as labor-augmenting innovations augmenting the "technology menu", (b) optimal factor-augmenting technology choice at the level of firms, and (c) "normalized" local and global CES production functions. In addition, by assuming that new ideas follow Weibull (rather than, say, Pareto) distributions, aggregate CES production is retained in our framework. Calibrating the model on US data, we perform numerical exercises allowing us to confirm that the interplay between endogenous growth channels indeed supports oscillatory convergence to the long-run growth path, and sometimes even to stable, self-sustaining (limit) cycles.

If agents are sufficiently patient (a low discount rate) and/or flexible in allocating consumption across time (a high elasticity of intertemporal substitution in consumption), the subsequent arrivals of both types of innovations can lead to limit cycle behavior via Hopf bifurcations. In such a case, irrespective of the initial conditions, the economy converges to a stable cyclical path where all trendless macroeconomic variables (such as the labor share) oscillate indefinitely around the steady state. Such oscillations have a predetermined frequency and amplitude.

Under our baseline calibration, however, the economy exhibits oscillatory convergence to a balanced growth path (BGP) rather than a limit cycle as such. Along this convergence path, the labor income share and other trendless variables of the model are subject to dampened oscillations, with a predetermined oscillation frequency and an exponentially decreasing magnitude.

1 Introduction

Labor's share of national income, once a focal point of debate from classical to post-war economists, has been rather neglected throughout the last half century. The recent surge of interest appears due to two reasons. First, the mounting evidence that, rather than being approximately stable, the labor share is quite volatile. Second, the observation that, in many countries, the labor share appears to have declined since the 1970s.

Both phenomena have been studied in isolation and different factors have been used to explain each of them. The contribution of this article is to bridge these two hitherto disconnected literatures, and to contribute a new way of thinking about labor-share movements. To our knowledge, we are the first to demonstrate that the labor's share of GDP exhibits pronounced *medium-run swings*. On top of that, we are the first to assess the extent to which an endogenous growth model can shed light on these features.

Looking at historical data for the US[1], we document that there has been a hump-shaped pattern in the labor share, coupled with marked medium-term volatility and high persistence. We also report its dynamic, medium-run correlation with other key macroeconomic variables (such as R&D expenditures). In total, these multi-faceted properties provide a motivation for setting up an endogenous, R&D-based growth model with endogenously determined cycles in factor income shares.

In contrast, existing models are largely silent on these issues. Models with Cobb-Douglas production (unit elasticity of substitution, neutral technical change) imply constant factor shares. And business-cycle models with variable markups generate shares that stabilize rapidly around a constant mean. Models endowed with a more general production specification (e.g., the neoclassical growth model with CES technology), on the other hand, do indicate a few critical tradeoffs; however, arguably the profession has not moved much beyond that.

Accordingly, conventional wisdom – i.e., that the labor share is largely driven by business cycles and is roughly stable over the long run – may (in both respects) be misleading. First, we show that most of the variance of the labor share lies *beyond* business-cycle frequencies. Therefore explaining labor-share movements with the aid of business-cycle mechanisms will only take us so far. Second, though necessarily stationary (i.e., lying within the unit interval), there is no compelling evidence that labor income shares are mean reverting (even in long-dated samples). This may reflect weaknesses of the relevant statistical tests, or it may point to an altogether different conclusion.

One suggestive way of reconciling these two aspects is to consider that labor income shares may be better characterized as being driven by a long cycle.[2] For instance in the US case, labor income share started from a low base in the late 1920s, stabilized somewhat

[1] Annual data on the labor share of income exists from 1929, and quarterly series from 1947q1. We analyze both series in the empirical sections of this paper.

[2] For long-dated historical analysis of factor developments, factor shares in value added and wealth distribution for several countries, see Piketty (2014).

at a higher value in the 1950-60s, then declined from the 1970s onwards. The 2012 value of the labor income share is essentially back to levels seen in the 1930s. Conceptually, that cycle may be either a long-lived dampened cycle, or even a stable, self-sustaining one.

If so, the challenge is to model and rationalize that cycle. We do so through the lens of an endogenous growth model with factor-augmenting R&D-based technical change.[3] The latter feature of the model is key. To illustrate: we know that under CES production, that, say, the capital income share, π, is by construction linked to the capital-output ratio, k/y and capital-augmenting technical progress, b: $\pi = f(k/y, b)$; the labor share is (in the simple case) $1 - \pi$. Given the strong mean reversion displayed by the former ratio, the processes underlying technological progress seem to be a relatively more promising starting point to examine cyclical features of the labor share.[4]

In our analysis, short-run frictions or exogenous shocks (otherwise popular in the literature to generate dynamics) are excluded.[5] Not only are these often weakly grounded in theory, they also pertain to business-cycle frequencies. Instead, focusing on R&D-led endogenous growth, we treat our model as a laboratory to assess mechanisms able to explain labor-share swings over the medium and long run. Calibrating the model on US data, we perform numerical exercises allowing us to confirm that the interplay between endogenous growth channels indeed supports oscillatory convergence to the long-run growth path, and sometimes even to stable, self-sustaining (limit) cycles.[6]

Moreover, the existing literature on endogenous cycles in growth models (albeit usually in systems of lower dimension than ours) has identified various supporting mechanisms: e.g., non concavities, adjustment costs, delay functions, information asymmetries, etc. In our model with gross complementarity between capital and labor in the aggregate CES production function, such oscillations appear endogenously as an outcome of the interplay between capital- and labor-augmenting R&D. If agents are sufficiently patient (a low discount rate) and/or flexible in allocating consumption across time (a high elasticity of intertemporal substitution in consumption), the subsequent arrivals of both types of innovations can lead to limit cycle behavior via Hopf bifurcations. In such a case, irrespective of the initial conditions, the economy converges to a stable cyclical path where all trendless macroeconomic variables (such as the labor share) oscillate indefinitely around the steady state. Such oscillations have a predetermined frequency and amplitude.

Under our baseline calibration, however, the economy exhibits oscillatory convergence

[3] Specifically, as far as we know, even though there exists a suite of endogenous growth models allowing for non-neutral technical change (Acemoglu, 2003), their implications for medium-to-long run swings in the labor share have not yet been analyzed. That economic activity may be subject to long waves of activity has proved influential following the works of Kondratieff and Schumpeter.

[4] Appendix B.2 provides some simple econometric motivation for this claim.

[5] Boldrin and Woodford (1990) discuss and assess different means to generate cycles in growth and business-cycle models.

[6] Articles, applications and surveys in this vein include Kaldor (1940), Goodwin (1951), Ryder and Heal (1973), Benhabib and Nishimura (1979), Dockner (1985), Boldrin and Woodford (1990), Feichtinger (1992), Faria and Andrade (1998).

to a balanced growth path (BGP) rather than a limit cycle as such. Along this convergence path, the labor income share and other trendless variables of the model are subject to dampened oscillations, with a predetermined oscillation frequency and an exponentially decreasing magnitude.

The structure of the paper is as follows. Section 2 documents the empirical evidence for medium- and long-run swings in the labor share. We find, using a variety of tools, that the labor income share has a complicated makeup: it is highly persistent and volatile, appears to be characterized by breaks and has a frequency decomposition skewed to the medium and long run. Section 3 then briefly reviews the empirical literature describing the labor share, and the associated theoretical contributions. In particular, we address the potential alternative explanations for long swings in the labor share, including its recent decline.

Section 4 contains the setup and solution of the theoretical model. We consider a non-scale model of endogenous R&D-based growth with (a) two R&D sectors, giving rise to capital- as well as labor-augmenting innovations augmenting the "technology menu", (b) optimal factor-augmenting technology choice at the level of firms, and (c) "normalized" local and global CES production functions. In addition, by assuming that new ideas follow Weibull (rather than, say, Pareto) distributions, aggregate CES production is retained in our framework.

Section 5 calibrates the model to US data and discusses the numerical results. We find that when compared with the socially optimal steady state, the decentralized labor share will in general be sub-optimal. We also investigate the effect perturbations in key parameters have on the labor share. We then formally consider the dynamic properties of the model in terms of oscillatory dynamics and stable limit cycles via Hopf bifurcations. Section 6 concludes.

2 Empirical Evidence for Long Swings in the Labor Share

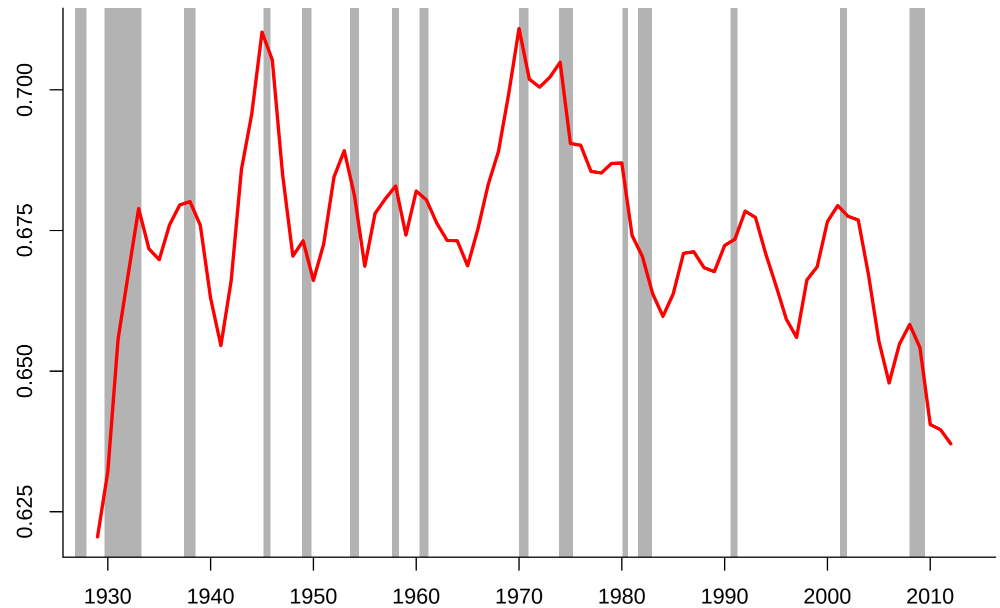
In this section we explore the medium-term properties of the US labor share. First, we highlight the extent and importance of its medium-run swings. Second, we formulate a range of associated stylized facts.

2.1 The Historical Time Series of the US Labor Share

The historical annual time series of US labor's share of GDP, spanning 1929–2012, is presented in **Figure 1**.[7]. Following Gollin (2002), we adjust the payroll share by proprietors' income (for details see Appendix A.1).

[7] In the following analysis, we will additionally use quarterly series, available for the period 1947q1-2013q1. This series can be viewed in Figure B.1 in the Appendix.

Figure 1: The Annual US Labor Share, 1929–2012



Note: Shaded areas represent recessions according to the NBER chronology overlaid at quarterly frequency. A table of summary statistics for the annual and quarterly labor income share is given in Appendix B.

The labor share series constructed here has all the properties identified in the earlier literature. First, it is counter-cyclical: it tends to rise during recessions (Young, 2004).[8] Second, there indeed has been a marked decline since 1970s (Karabarbounis and Neiman, 2014). However, our use of long time series make us appreciate that this is only part of the story: in fact, before the labor share began this decline, it showed an upward tendency. Indeed, when examined over the entire period, the historical series arguably looks part of a long cycle.[9]

This "hump-shaped" pattern has some interesting implications. According to Kaldor's (1961) stylized facts, factor shares should be broadly stable over time (or at least mean reverting). Formal stationarity tests, though, are inconclusive (see **Table B.2**, Appendix B). This may reflect the low power of tests in finite samples to distinguish between a non-stationary and near-stationary process, and/or the distorting presence of a structural break(s), Perron (1989). It may also reflect the absence of an incomplete cycle in the observed data. We now examine these points in more depth below.

[8] A regression of the quarterly labor income share on a constant and an NBER recession dummy; a constant, an AR(1) term and a recession dummy; and a constant, an AR(1) term, a linear time trend and recession dummy, respectively, yield the parameters on the recession dummy equal to 0.0061; 0.0044**; 0.0037**; 0.0040**. Superscript ** indicates statistical significance at 5%.

[9] Factor shares do not, to repeat the point made in the Introduction, appear to have inherited this type of pattern from the capital-output ratio, which is one of its fundamental determinants. See Figure B.2 in appendix B.

2.2 Persistence

To scrutinize the degree of persistence of the labor share, we assume that it follows an AR(1) process:

$$y_t = \mu + \rho_y y_{t-1} + \varepsilon_{y,t}, \tag{1}$$

where the drift term μ captures the long-run mean, $\mu/(1-\rho_y)$ with $\rho_y \neq 1$, and $\varepsilon_{y,t} \sim \mathcal{N}(0, \sigma_y)$. Our interest focuses on both the value and stability of ρ_y (the persistence parameter). **Table 1** demonstrates that the labor share is a highly persistent, slowly-adjusting series (with ρ_y around 0.8 and 0.95 for annual and quarterly series respectively). This is robust to the inclusion of a linear or quadratic trend.[10]

Further, in around 80% of the cases, there is evidence for one or more unknown structural parameter breakpoints using the Quandt-Andrews breakpoint test. This in turn will be validated by three further pieces of evidence: (a) rolling parameter estimates, (b) structural break tests, and (c) Markov switching. The latter can be seen as a indicator of persistence in the presence of complex, non-linear breaks.

Table 1: AR(1) Model Estimates for the Labor Share

	ANNUAL SERIES			QUARTERLY SERIES		
	(1)	(2)	(3)	(1)	(2)	(3)
$\hat{\mu}$	0.108***	0.130***	0.176***	0.017	0.029**	0.053***
$\hat{\rho_y}$	0.840***	0.814***	0.739***	0.975***	0.958***	0.920***
$\hat{\beta}_1 \cdot 100$		−0.010**	0.020		−0.001	0.003**
$\hat{\beta}_2 \cdot 1000$			−0.004*			$-1.650e-5$***
$\hat{\sigma_y}$	0.008	0.008	0.007	0.004	0.004	0.004
R^2_{adj}	0.754	0.775	0.781	0.933	0.934	0.935
$\rho_y = 1$	[0.002]	[0.001]	[0.000]	[0.116]	[0.029]	[0.001]
	Quandt-Andrews unknown breakpoint test					
Max LR F-stat	[0.026]	[0.175]	[0.027]	[0.007]	[0.002]	[0.005]
Exp LR F-stat	[0.006]	[0.201]	[0.402]	[0.009]	[0.005]	[0.042]
Ave LR F-stat	[0.001]	[0.099]	[0.295]	[0.011]	[0.001]	[0.017]

Note: Superscripts ***, ** and * denote the rejection of null about parameter's insignificance at 1%, 5% and 10% significance level, respectively. Probability values in squared brackets. For the break-point tests, we employ a standard 15% sample trimming, and use probability values from Hansen (1997).
Specifications:
(1): $y_t = \mu + \rho_y y_{t-1} + \varepsilon_{y,t}$
(2): $y_t = \mu + \rho_y y_{t-1} + \beta_1 t + \varepsilon_{y,t}$
(3): $y_t = \mu + \rho_y y_{t-1} + \beta_1 t + \beta_2 t^2 + \varepsilon_{y,t}$

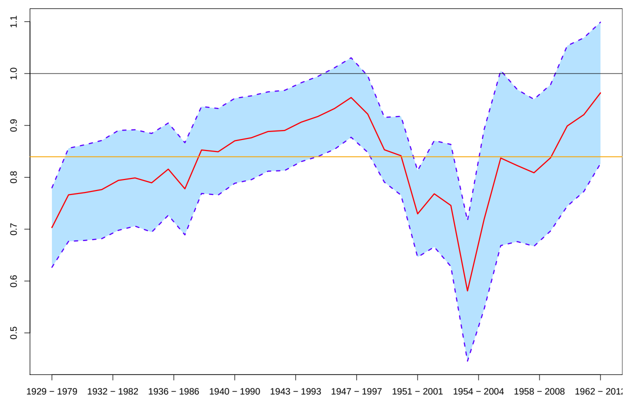
[10] Although, naturally, these alternative forms relax the assumption about the uniqueness of the labor share's equilibrium level.

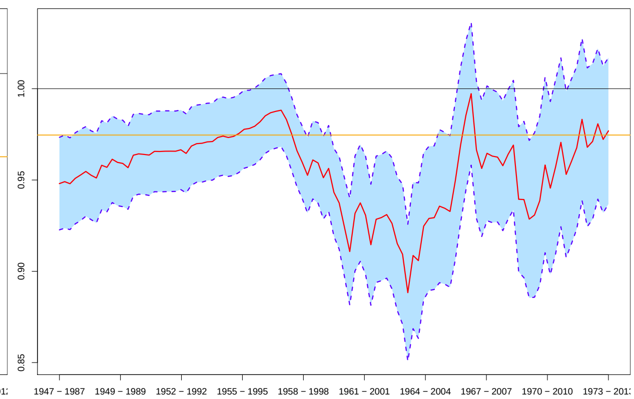
2.3 Rolling Window Estimation of the AR Model

On point (a) above, **Figure 2** demonstrates that the auto-regression exhibits substantial variation between different sub-samples: from 0.61 to 0.96 and from 0.88 to 1.00 for annual and quarterly data, respectively.[11] The trajectory, though, is similar for both: U-shaped and with the lowest $\hat{\rho_y}$ occurring at more recent windows, e.g. ending around 2003. Apart from the adjustment parameters, the standard deviation of shocks driving the labor share $\hat{\sigma_y}$ also varies strongly and, independently of data frequency, is also characterized by a U-shaped trajectory. Here, the lowest $\hat{\sigma_y}$ is about 35% lower than the full sample estimates and is reached at the same window as in the $\hat{\rho_y}$ case.

Figure 2: Rolling Window Estimates of ρ_y

a: annual series (50Y) **b: quarterly series (40Y)**





Note: In addition to the point estimates plus confidence intervals, two reference lines are included at unity, and the sample mean of ρ_y.

2.4 Structural Breaks

Regarding the dating of structural breaks (point (b)), the results of the Bai and Perron (2003) procedure are reported in **Table 2**. The number of breaks is selected using the BIC criterion with an upper bound of five. Irrespective of data frequency and the assumption on the data-generating process, two breakpoints are robustly identified: in the second half of the 1960s, and in the first half of the 1980s. These dates delineate the successive periods characterized by an upward tendency, a short period of erratic fluctuations, and a downward tendency.

The statistical evidence of at least two structural breaks clearly indicates a nontrivial dynamic behavior of the labor share. In the next step, we run Markov-switching regressions to explore potential nonlinearities in the labor share's dynamic behavior.

[11] In order to check if the degree of persistence is roughly constant, the length of window is fixed at 50 and 40 years for the annual and quarterly series, respectively.

Table 2: Structural Breaks in the Labor Share

	ANNUAL SERIES			QUARTERLY SERIES		
Specification	(1)	(2)	(3)	(1)	(2)	(3)
breakpoints	1942	1946	1942	1967Q4	1968Q1	1960Q3
	(1938,1953)	(1945,1956)	(1941,1943)	(1967Q1,1968Q2)	(1967Q3,1968Q2)	(1959Q1,1960Q2)
	1967	1968	1969	1980Q3	1986Q1	1969Q4
	(1963,1974)	(1967,1969)	(1968,1970)	(1980Q2,1981Q1)	(1985Q4,1986Q3)	(1969Q3,1970Q1)
	1980	1985	1994	2003Q2	1999Q4	1981Q2
	(1978,1981)	(1984,1987)	(1993,1995)	(2002Q1,2003Q4)	(1998Q4,2000Q1)	(1980Q4,1981Q4)
		1999				1992Q1
		(1995,2000)				(1991Q1,1992Q2)
						2001Q4
						(2001Q3,2002Q1)
breaks	3	4	3	3	3	5

Note: Specifications: (1) - only mean; (2) - linear trend; (3) - quadratic trend. Confidence intervals in terms of dates reported in parenthesis.

2.5 Markov-Switching (MS) Results

Although MS models are typically applied in the business-cycle literature (Hamilton, 1989), e.g. to date recessions or identify asymmetrical business cycles, they might also be useful for capturing long-lasting regime changes. While moving window estimation already allowed us to identify a U-shaped pattern in the degree of persistence of the series, Markov-switching AR models are a valid extension of such analysis because here, possible changes in the variance are introduced directly into the full-sample estimation procedure. This method is therefore useful for verifying the existence of endogenous cycles driving the labor share.

Formally, the two-regime Markov-switching AR(1) model is,

$$y_t = \mu_s + \rho_{y,s} y_{t-1} + \varepsilon_{y,t}, \qquad \varepsilon_{y,t} \sim \mathcal{N}(0, \sigma_{y,s}) \tag{2}$$

where $s_t = 1, 2$ is the latent variable indicating the current regime. We consider four variants of the model: with constant ρ_y and μ (specification no. 1), regime-specific μ (2), regime-specific ρ_y (3), and regime-specific μ and ρ_y (4). All variants assume regime-specific variance of shocks.

The estimates are presented in **Table 3**, for both the raw and de-trended (quarterly) labor income share. It turns out that the differences in the error variance between regimes are robust to the model specification. The more volatile regime is characterized with a 75% higher standard deviation of shocks. Once we relax the assumption of a constant adjustment parameter, we might identify two regimes: one with a lower variance of shocks and higher persistence, and a second one that is less persistent but more volatile.

Table 3: MS-AR(1) Model Estimates for the Quarterly Labor Share

	RAW SERIES				DE-TRENDED SERIES			
	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)
$\hat{\mu}$ or $\hat{\mu}_1$	0.017	0.020**	0.021	0.033	-0.0001	-0.0007	-0.0001	-0.0007^{*}
$\hat{\mu}_2$		0.021**		0.007		0.0003		0.0004**
$\hat{\rho}_y$ or $\hat{\rho}_{y,1}$	0.975***	0.969***	0.968***	0.950***	0.920***	0.920***	0.721***	0.733***
$\hat{\rho}_{y,2}$			0.969***	0.991***			0.981***	0.983***
$\hat{\sigma}_{y,1}$	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005
$\hat{\sigma}_{y,2}$	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003
$\hat{\sigma}_{y,1}/\hat{\sigma}_{y,2}$	1.828	1.806	1.801	1.800	1.744	1.770	1.629	1.660
$\hat{p}_{1,1}$	0.968	0.965	0.965	0.966	0.951	0.951	0.978	0.977
$\hat{p}_{2,2}$	0.967	0.971	0.972	0.972	0.969	0.958	0.981	0.980
$\mathcal{D}_1$	7.812	7.143	7.143	7.353	5.102	5.102	11.364	10.870
$\mathcal{D}_2$	7.576	8.621	8.929	8.929	8.065	5.952	13.158	12.500
$n_1/(n_1+n_2)$	0.498	0.434	0.434	0.434	0.332	0.457	0.453	0.472

Note: Superscripts ***, ** and * denote significance at 1%, 5% and 10% significance level, respectively. $\mathcal{D}_i$, the (annual) duration of regime i, is calculated as $\frac{1}{4}(1-\hat{p}_{i,i})^{-1}$ and $n_1/(n_1+n_2)$ is a fraction of observations classified to the first regime.

Viewed from the medium-term perspective, the elements of the transition matrix are of particular interest. We note that $\hat{p}_{1,1}$ (i.e., the probability of staying in regime 1) and $\hat{p}_{2,2}$ are similar across all considered specifications. Accordingly, there is no strong evidence for asymmetric transition – as opposed to the business-cycle literature which typically finds that the boom regime dominates the contraction regime in terms of duration. Here, the expected duration in each regime is around $8-12$ years, again lending weight to the medium-run interpretation of income shares.

Finally, using smoothed probabilities (see **Figure B.3**, Appendix B), we identify two periods with a relatively lower variance of shocks, and higher persistence: 1960-1975 and 1980-2000. These match very well the earlier identified breaks and common US-growth narratives (respectively, the high post-war growth and the 'Great Moderation").

2.6 Spectral Analysis

The previous evidence of nonlinear dynamics is suggestive of the importance of the medium-run variation in the labor share. To verify the actual magnitude and frequency of medium-run swings, we turn to spectral techniques. Accordingly, the variation in the time series shall be split into three ranges in the frequency domain: high-frequency (with periodicity below 8 years), medium-frequency (periodicity between 8 and 50 years) and low-frequency oscillations (periodicity above 50 years). Note that the high-frequency component includes mainly business-cycle fluctuations (and noise). Furthermore, for spectral density estimation, the data should be demeaned or de-trended.[12] Having no priors, three variants of de-trending (demeaning) are considered.

Table 4 presents the estimated share of specific types of fluctuations in the total variance of the quarterly labor share series. In the case of demeaned series, medium-frequency fluctuations are responsible for $38 - 48\%$ of total volatility and the cycles mapped into the low-frequency pass are almost just as important. As expected, de-trending the labor share series limits the contribution of low-frequency oscillations in the overall variance, and medium-term fluctuations become more important instead, with their share about 50% and 70% for the series de-trended by a linear and quadratic trend, respectively.

Table 4: Share of Specific Frequencies in Total Variance of the Labor Share (In %)

	ANNUAL			QUARTERLY		
PERIODICITY (IN YEARS)	≥ 50	8-50	≤ 8	≥ 50	8-50	≤ 8
Excluding the mean	31.5	48.3	20.2	42.8	37.9	19.3
Excluding a linear trend	28.3	48.9	22.7	29.8	52.4	17.8
Excluding a quadratic trend	0.2	68.1	31.7	0.4	72.6	27.0

Note: the shares have been calculated using periodogram estimates.

To sum up, spectral analysis provides us with an additional argument for the importance of medium-run swings in the labor share. Irrespectively of the de-trending strategy, the share of medium-term fluctuations in the overall variance is meaningful and at least two times higher than the fraction linked to the short-run oscillations.

Looking at the periodogram estimates for both annual and quarterly data, we also note that there are two dominant frequencies of fluctuations in the US labor share: (a) medium-term cycles lasting around 30 years, and (b) the long-run stochastic trend, whose length reaches beyond the 80 years mark. As opposed to business-cycle models, the mechanisms present in our model are able to generate swings of either such frequency.

2.7 Stylized Facts about the Labor Share's Medium-Term Fluctuations

Thus the US labor share exhibits pronounced medium-term swings. To formulate a range of stylized facts about their properties, we follow Comin and Gertler (2006). Using their definition, medium-term fluctuations are identified here as all cycles with periodicity between 8 and 50 years.[13]

[12] The estimation on the spectral density is sensitive to the existence of both a unit root and a deterministic component in the series.

[13] More precisely, Comin and Gertler (2006) define medium-term business cycles as all fluctuations between 2 and 50 years, and then they divide such cycles into two components: the low-frequency component (with periodicity between 8 and 50 years) and the high-frequency component (with periodicity below 8 years). Note that the second one includes mainly the business-cycle fluctuations which are excluded from the current study.

The choice of method for extracting the medium-term component from the data is mostly determined by the frequency domain in question. Following earlier work on medium-term cycles (e.g., Comin and Gertler (2006), Correa-López and de Blas (2012)), we apply the Christiano and Fitzgerald (2003) (CF) approximation of the ideal band-pass filter.[14] The general strategy of isolating the medium-term component is the following. Because from the purely statistical point of view based on the results of stationarity tests, both the labor share and virtually all its correlates in question are non-stationary, we transform our data into log differences and then apply the band-pass (CF) filter. Next, we cumulate the filtered data and demean. This increases filter efficiency as compared to applying the filtering procedure directly to log levels. The extracted series represent percentage deviations from the long-run stochastic trend.[15]

The medium-term component extracted from the labor share is depicted in **Figure 3**. We see that the medium-frequency cyclical component is responsible for a significant part of the overall volatility of the series and that it has an important contribution to the scale of deviation from long run trend at the turning points. Although isolating the medium- and high-frequency cycles reduces the volatility substantially, the remaining smoothed long-run trend is still hump-shaped (with a peak around the early 1960s).

Akin to the real business cycle literature, we report the main features of the medium-term component of the labor share using moments of the filtered series: volatility (standard deviation), persistence (first-order autocorrelation) and co-movement (cross-correlation function) with selected other macroeconomic variables. Apart from the point estimates of the selected moments, we report also confidence intervals.[16] The general statistics are reported in **Table 5**.

Based on this, we formulate the following stylized facts:

(a) The medium-term component of the labor share is highly persistent (0.94-0.99).

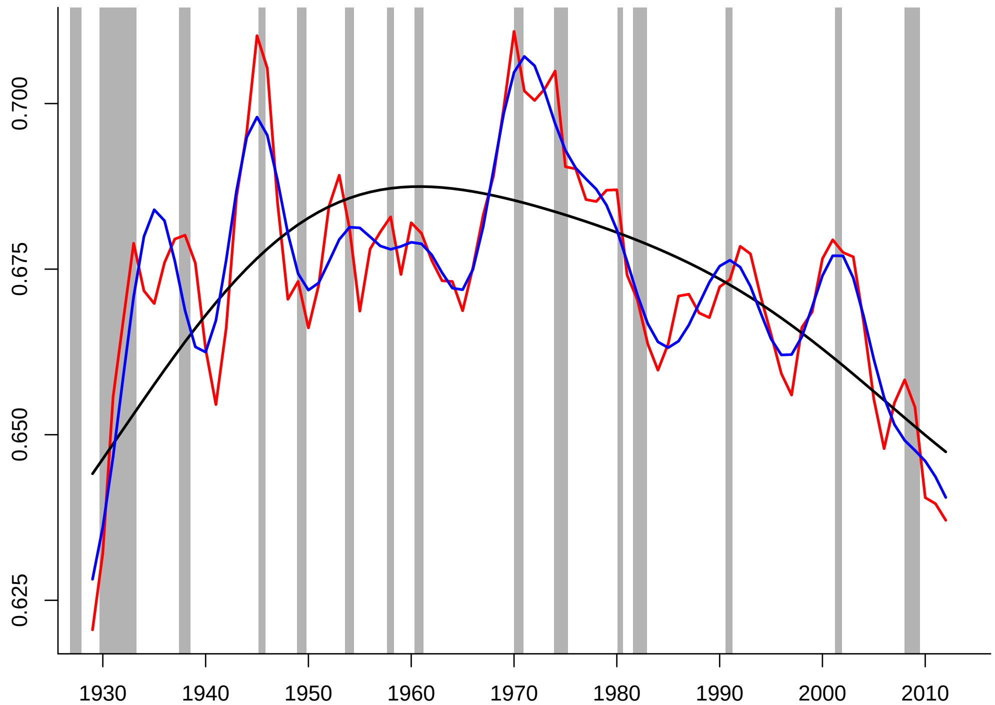
(b) The labor share fluctuates substantially in the medium-term frequencies. The volatility of labor share is about 30% of the volatility of output.

[14] As a low-pass filter, the well-known Hodrick-Prescott (HP) filter would have to be used twice here to deliver an appropriate frequency band. One could also use the Baxter-King (BK) filter. There are three advantages of the CF procedure, however. First, it is a better approximation of the ideal band-pass filter than the twice convoluted HP filter. Second, applying the BK approximation incurs a loss in the number observations in the filtered series. Third, the fundamental assumption of the CF filter is the fact that data are generated by a random walk. Therefore, in our case the Christiano-Fitzgerald procedure is a more plausible way of extracting the low-frequency component.

[15] Interestingly, the results presented in this section are robust to the filtering strategy. For the labor share, general features are virtually the same when the CF procedure is applied to demeaned or de-trended data. For the remaining series, removing the linear trend leads to the similar conclusions.

[16] The confidence intervals have been computed using a bootstrap. More precisely, for a given moment we run a given-sized sample with replacement. The CIs are the 0.025 and 0.975 quantiles of the obtained simulated distributions. In the table we use 5000 replications; a larger number did not change the results.

Figure 3: The Medium-Term Component and the Long-Term Stochastic Trend of the Annual US Labor Share



Note: the red, blue and black lines represent the raw series, the medium-term component and the long-run trend, respectively.

Table 5: Features of Labor Share's Medium-Term Component

	σ_{LS_t}	$\sigma_{LS_t}/\sigma_{GDP_t}$	$\rho_{LS_t,LS_{t-1}}$	ρ_{LS_t,GDP_t}
ANNUAL SERIES	1.450	0.290	0.940	0.480
	(1.230,1.640)		(0.900,0.970)	(0.270,0.650)
QUARTERLY SERIES	1.450	0.300	0.996	0.540
	(1.330,1.540)		(0.995,0.996)	(0.470,0.620)

Note: σ_{LS_t} and $\sigma_{LS_t}/\sigma_{GDP_t}$ denote the volatility in absolute terms (percentage deviation from the long-run trend) and relative terms (as a ratio to the volatility of GDP). $\rho_{LS_t,LS_{t-1}}$ and ρ_{LS_t,GDP_t} stand for the first-order autocorrelation and contemporaneous co-movement with output, respectively.

(c) The labor share is positively correlated with output. Even if the estimates of contemporaneous co-movement are not so high (around 0.5), they remain in stark contrast to the negative correlation repeatedly reported for the short-run component.

The pro-cyclicality of the medium-term component of the labor share is probably the most surprising stylized fact. Moreover, the phase shift between cyclical components of those variables suggests that highest correlation is observed if the labor share is lagged by two years (though the difference of that lagged correlation from the contemporaneous one is not statistically significant).

A wider range of statistics describing the labor share's co-movement with main macroeconomic variables is presented in **Table 6**. Virtually all pro-cyclical variables in the medium-term business cycles, for instance investment I_t, consumption C_t, hours worked, and employment, are also positively correlated with the labor share.[17]

Table 6: Selected Labor Share Cross-Correlations

	Correlation with the Labor Share					Medium-Term Characteristics			
	ρ_{LS_t,x_t}	$\rho_{LS_{t+\tau},x_t}$	τ	$\rho_{LS_{t+\tau},x_t}$	τ	σ_{x_t}	$\sigma_{x_t}/\sigma_{y_t}$	$\rho_{x_t,x_{t-1}}$	ρ_{y_t,x_t}
$K_t^{private}/Y_t$	-0.18	0.75	7	-0.85	-5	6.17	1.24	0.96	-0.03
	(-0.38,0.04)	(0.64,0.83)		(-0.91,-0.77)		(5.39,6.84)		(-0.32,0.27)	(-0.32,0.27)
C_t	0.53	0.57	-1	-0.32	-10	3.44	0.69	0.96	0.87
	(0.37,0.68)	(0.41,0.69)		(-0.57,-0.05)		(2.97,3.91)		(0.94,0.97)	(0.8,0.92)
I_t	0.46	0.59	-3	-0.42	-10	11.83	2.39	0.94	0.51
	(0.27,0.64)	(0.46,0.7)		(-0.61,-0.18)		(9.94,13.55)		(0.91,0.97)	(0.34,0.66)
G_t	0.30	0.54	-3	-0.29	5	7.40	2.49	0.86	-0.02
	(0.09,0.49)	(0.29,0.73)		(-0.47,-0.06)		(5.96,8.68)		(0.77,0.92)	(-0.23,0.2)
C_t/K_t	0.15	0.78	-6	-0.77	7	6.15	1.24	0.97	0.14
	(-0.07,0.35)	(0.69,0.86)		(-0.85,-0.66)		(5.26,6.93)		(0.96,0.98)	(-0.14,0.39)
C_t/Y_t	-0.02	0.25	-10	-0.41	7	1.61	0.32	0.90	0.26
	(-0.22,0.18)	(-0.01,0.46)		(-0.59,-0.19)		(1.26,1.99)		(0.82,0.95)	(0.07,0.49)
I_t/Y_t	0.11	0.43	5	-0.55	-10	10.6	2.14	0.94	0.21
	(-0.05,0.28)	(0.22,0.61)		(-0.68,-0.36)		(9.09,11.92)		(0.91,0.96)	(-0.04,0.43)
$Employment_t$	0.42	0.62	-2	-0.47	-9	2.15	0.43	0.88	0.48
	(0.26,0.56)	(0.46,0.75)		(-0.65,-0.23)		(1.86,2.41)		(0.82,0.93)	(0.28,0.63)
$hours_t$	0.34	0.80	-3	-0.27	10	2.62	0.53	0.9	0.52
	(0.15,0.51)	(0.7,0.87)		(-0.54,0.03)		(2.22,2.98)		(0.85,0.94)	(0.33,0.68)
$LaborProd_t$	0.42	0.49	2	-0.25	-10	3.97	0.8	0.98	0.83
	(0.23,0.58)	(0.27,0.67)		(-0.51,0.02)		(3.59,4.34)		(0.97,0.99)	(0.75,0.89)
$R\&D_t/GDP_t$	-0.3	0.73	-10	-0.62	3	12.29	2.48	0.96	0.13
	(-0.47,-0.12)	(0.59,0.84)		(-0.73,-0.5)		(10.15,14.34)		(0.93,0.98)	(-0.05,0.3)
$R\&D_t$	-0.07	0.59	-9	-0.45	4	13.85	2.79	0.97	0.42
	(-0.28,0.15)	(0.38,0.76)		(-0.62,-0.26)		(11.54,16.05)		(0.94,0.98)	(0.23,0.59)

Note: ρ_{y_t,x_t} and ρ_{LS_t,x_t} denote the contemporaneous cross-correlation for series x_t with output and the labor share. $\rho_{LS_{t+\tau},x_t}$ reflects to the correlation of variable x_t with labor share lagged by τ period. For the labor share the highest and the lowest cross-correlation with each series are reported. $\rho_{x_t,x_{t-1}}$ and σ_{x_t} denote the first-order autocorrelation and standard deviation from the long-run trend, respectively.

Our key observations here are as follows. First, although the contemporaneous co-movement of the labor share with R&D expenditures is not statistically significant, it becomes negative and significant once we lag the labor share by 3-4 years. This indicates that

[17] We performed these correlations for a far larger set of variables than shown here. Details available upon request.

bursts in R&D activity can be viewed as a leading indicator for downward swings in the labor share 3-4 years later. A similar pattern is identified for the consumption-capital ratio: periods when consumption is relatively high are followed (in around 7 years) by similar downward swings. On the other hand, the opposite is found for the capital-output ratio: periods of capital-intensive production are followed (in around 7 years) by periods where the labor share rises.

Finally, on the skill-premium series, in the medium-term frequency range, the cyclical component of w_t^S/w_t^U is quite volatile and persistent. Furthermore, the contemporaneous co-movement with output is strongly negative. Interestingly, the pairwise correlation with the pro-cyclical labor share is just at the border of statistical significance.

To conclude, the empirical evidence on the dynamic behavior of the medium-term component of the US labor income share points out at its high persistence, substantial volatility, and interesting patterns of correlation with other macroeconomic variables, including R&D and capital deepening. In the following section, we take a brief look at the literature on describing movements in the labor income share.

3 Associated Literature

The literature describing movements in the labor share is diverse, encompassing different theoretical and empirical perspectives. A full review is beyond our purpose, but we can touch on some common and related themes.

The recent decline in labor income share poses a particular challenge to theory. The usual macroeconomic paradigm of Cobb-Douglas production coupled with isoelastic demand (leading to constant markups) leaves no room for the prolonged swings in factor shares observed in the data. The literature therefore tries to explain this phenomenon as departures from that benchmark.

For instance, movements in labor share have often been seen as reflecting transitory capital augmenting technical change acting alongside the more standard labor-augmenting variety, (Acemoglu, 2003; Bentolila and Saint-Paul, 2003; Jones, 2005b; Klump, McAdam, and Willman, 2007). Given that capital and labor are gross complements, this pattern ensures asymptotically stable income shares while allowing for fluctuations in the transition.

At a less aggregate level, there is also the role of skill-biased technical change (SBTC). This is defined as a change in the production technology that favors "skilled" over "unskilled" labor by increasing the former's relative productivity and, therefore, relative demand. SBTC may operate through different channels, such as specific combinations of factor complementarity or directed technical change or both, see Acemoglu (2002) for a discussion. Notwithstanding, such developments will necessarily impact income shares both between labor types and between labor and capital (e.g., depending on what proportion of labor is high skilled).

At an even more disaggregate level, structural transformation within the economy may also help explain trends in factor income shares, see Kongsamut, Rebelo, and Xie (2001), de Serres, Scarpetta, and de la Maisonneuve (2002), Ngai and Pissarides (2007), Acemoglu and Guerrieri (2008), Buera and Kaboski (2012). Think for example of the rise in output shares of Manufacturing and Services, and the decline of Agriculture. Such shifts may impact income shares depending on the skills and bargaining power of the affected labor; increasing female labor force participation due to substituting home production with market services (Buera and Kaboski, 2012); the overall trend in mark-ups resulting from the sectoral changes (McAdam and Willman, 2004); the changing patterns of firm size and age (Kyyrä and Maliranta, 2008); the rise of offshoring (Elsby, Hobijn, and Sahin, 2013), and so on.

These explanations all have pros and cons of one sort or another. Those based on factor-augmenting technical change, sectoral R&D and endogenous growth are ones that we are drawn to, largely for their simplicity and intuition. Accordingly, in the following sections we outline the model to be used.

4 Model

We consider a non-scale model of fully endogenous R&D-based growth with (a) two R&D sectors, giving rise to capital- as well as labor-augmenting innovations augmenting the technology menu (Acemoglu, 2003), (b) optimal factor-augmenting technology choice at the level of firms, following Jones (2005b), and (c) normalized local and global CES production functions (Klump and de La Grandville, 2000). Before that, we strive to put our contribution in the context of the existing literature.

4.1 Endogenous Technology Choice in the Literature

Our starting point is Jones (2005b) who, building on Houthakker (1955), argued that the aggregate or "global" production function (GPF) can be viewed as the upper envelope of "local" production functions (LPF). The latter should exhibit low factor substitution reflecting the quasi fixity of ideas.[18] However, assuming that new techniques are independently and identically drawn from a Pareto distribution, Jones (2005b) demonstrates that the GPF actually tends to Cobb-Douglas (with technical change asymptotically labor-augmenting). The Pareto form, moreover, is popular since, given its heavy-tailed distribution, it matches many economic phenomena (e.g., firm and city size, stock returns). Plus it embodies a proportionality factor, i.e., $\mathbb{E}[x|x \geq \tilde{x}] \propto \tilde{x}$ which is intuitive in the modelling of ideas.

[18]Often the LPF elasticity is motivated as Leontief. But perfect complementarity is a strong assumption with the counter-factual implication that output shares of capital and labor approach one-half, and is also usually ruled out given its knife-edge implications for growth and optimal savings, (de La Grandville, 2009).

Notwithstanding, there is no *overwhelming* reason to choose this particular distribution, or to discard plausible alternatives. Indeed in our context, the assumption is not without its drawbacks. First, a unitary elasticity of substitution is highly counter-factual (Klump, McAdam, and Willman, 2012). Thus to build a theory on the back of a parameter value for which there is limited empirical support may be regarded as unsatisfying. Likewise, although necessarily bounded, the labor share has been shown to fluctuate substantially with very long swings. Whilst aggregation towards the global Cobb-Douglas form does not preclude such fluctuations, it still leaves a gap in our knowledge of how factor income shares behave in the transition. Is that dynamic monotonic or oscillatory, or even unique? Is its cycle length empirically plausible? In addition, one (so far neglected) possibility is that income shares have a self-sustaining dynamic; thus even in the long run, without exogenous shocks, they continue to fluctuate around their steady state value.

As far as the specific issue of Pareto distributions is considered, Growiec (2008) has demonstrated, however, that if new techniques are instead independently Weibull-distributed then the GPF is not Cobb-Douglas, but rather CES. The Weibull form, moreover, is appealing from a number of perspectives. Assuming that factor-augmenting technologies are inherently complex and consist of a large number of complementary components, the Weibull distribution should approximate the true productivity distribution better than anything else, including the Pareto (Growiec, 2013). The argument is based on the extreme value property of the Weibull (de Haan and Ferreira, 2006): if one takes the minimum of n independent draws from some (sufficiently well-behaved) distribution, then as $n \to \infty$, this minimum will converge in distribution to the Weibull.

Taking the minimum corresponds to the case of complex technologies consisting of complementary components (e.g. Kremer, 1993) whose productivity is determined by that of their "weakest link". The same complementarity requirement, coupled with the assumption of limited substitution possibilities along the local production, implies also that factors should be gross complements along the global CES production function. Accordingly with this, we can maintain the more empirically relevant CES global function.

4.2 Local and Global Technology

Assume that the local production function takes the normalized CES or normalized Leontief form (de La Grandville, 1989; Klump and de La Grandville, 2000):

$$Y = Y_0 \left(\pi_0 \left(\frac{bK}{b_0 K_0} \right)^{\theta} + (1 - \pi_0) \left(\frac{aL_Y}{a_0 L_{Y0}} \right)^{\theta} \right)^{\frac{1}{\theta}}, \tag{3}$$

In the normalization procedure of the CES LPFs, benchmark values are assigned not only to output, capital and labor (Y_0, K_0, L_0), but also to the benchmark technology (b_0, a_0).[19] In

[19]Normalization essentially implies representing the production function and factor demands in consistent indexed number form. Without normalization, it can be shown that the production parameters have no

the following derivations, this benchmark technology will be identified with the *optimal* technology at time t_0. Factors are assumed to be gross complements along the LPF (i.e., $\theta < 0 \Leftrightarrow \sigma_{LPF} < 1$). Under the assumption that the representative firm operates in a perfectly competitive environment, which is possible given constant returns, the capital share equals,[20]

$$\pi = \frac{\pi_0 \left(\frac{bK}{b_0 K_0}\right)^{\theta}}{\pi_0 \left(\frac{bK}{b_0 K_0}\right)^{\theta} + (1-\pi_0)\left(\frac{aL_Y}{a_0 L_{Y0}}\right)^{\theta}} = \pi_0 \left(\frac{b}{b_0}\right)^{\theta} \left(\frac{KY_0}{K_0 Y}\right)^{\theta}. \tag{4}$$

and the labor share amounts to $1-\pi$. Thus the capital share is determined by capital augmenting technology and the capital-output ratio.

The assumption that there is limited substitutability along the LPF is consistent with the "recipe" interpretation of particular production techniques (where the LPF is viewed as a list of instructions on how to transform inputs into output, that must be followed as closely as possible, cf. Jones (2005b)). It is thus clearly preferred over the alternative cases of $\theta \to 0$ (Cobb–Douglas LPFs) or $\theta \in (0,1)$ (gross factor substitutability along the LPF). These cases being respectively counter-factual and theoretically anomalous.[21,22]

The "technology menu" specified in the (a,b) space is given by,

$$H(a,b) = \left(\frac{a}{\lambda_a}\right)^{\alpha} + \left(\frac{b}{\lambda_b}\right)^{\alpha} = N, \qquad \lambda_a, \lambda_b, \alpha, N > 0. \tag{5}$$

The menu is thus downward sloping in (a,b), capturing the trade-off between the available unit factor productivities (UFPs) of capital and labor. The technology menu can be understood as a *contour line of the cumulative distribution function* of the joint bivariate distribution of capital- and labor-augmenting ideas ($\tilde{b}$ and $\tilde{a}$, respectively). Under independence of both dimensions (so that marginal distributions of $\tilde{b}$ and $\tilde{a}$ are simply multiplied by one another), equation (5) is obtained iff the marginal distributions are Weibull with "shape" parameter $\alpha > 0$:

$$P(\tilde{a} > a) = e^{-\left(\frac{a}{\lambda_a}\right)^{\alpha}}, \qquad P(\tilde{b} > b) = e^{-\left(\frac{b}{\lambda_b}\right)^{\alpha}}, \tag{6}$$

economic interpretation since they are dependent on the normalization point and on the elasticity of substitution itself. This feature significantly undermines estimation and comparative-static exercises, among other things. See de La Grandville (1989) and Klump and de La Grandville (2000) for seminal contributions and Klump, McAdam, and Willman (2012) for a survey. Growiec (2013) and Leon-Ledesma and Satchi (2011) provide micro-foundations for the normalized CES production framework.

[20]We confine ourselves to constant-returns production functions, consistent with much of the aggregate evidence, e.g., Basu and Fernald (1997).

[21]Chirinko (2008) and León-Ledesma, McAdam, and Willman (2010) tabulate a number of empirical production studies over many different historical periods and countries, and find little evidence for unitary or above unitary forms. The most famous Cobb-Douglas finding was Berndt's (1976), although it should be recalled that his result concerned only the US manufacturing sector.

[22]Solow (1956) showed in the neoclassical growth model that a CES function with an elasticity of substitution greater than one can generate sustained growth (even without technical progress). de La Grandville (2009) discusses the precise threshold conditions associated to this perpetual-growth scenario.

for $a, b > 0$. Under such parametrization, we have $P(\tilde{a} > a, \tilde{b} > b) = e^{-\left(\frac{a}{\lambda_a}\right)^{\alpha} - \left(\frac{b}{\lambda_b}\right)^{\alpha}}$, and thus $N = -\ln P(\tilde{a} > a, \tilde{b} > b) > 0$. We assume N to be constant, and for λ_a and λ_b to grow as an outcome of factor-augmenting R&D.

Firms choose the technology pair (a, b) subject to the current technology menu, such that their profit is maximized:[23,24]

$$\max_{a,b} \left\{ Y_0 \left(\pi_0 \left(\frac{bK}{b_0 K_0} \right)^{\theta} + (1-\pi_0) \left(\frac{a L_Y}{a_0 L_{Y0}} \right)^{\theta} \right)^{\frac{1}{\theta}} \right\} \qquad s.t. \qquad \left(\frac{a}{\lambda_a} \right)^{\alpha} + \left(\frac{b}{\lambda_b} \right)^{\alpha} = N. \tag{7}$$

It is easily verified that at time $t = t_0$ (the point of normalization), the optimal choice is then,

$$a_0^* = (N(1-\pi_0))^{\frac{1}{\alpha}} \lambda_{a0}, \qquad b_0^* = (N\pi_0)^{\frac{1}{\alpha}} \lambda_{b0}, \tag{8}$$

The above values of a_0^* and b_0^* will be used as a_0 and b_0 in the normalization at the local level in all subsequent derivations. For any other $t \neq t_0$, the optimal technology choices are:

$$\left(\frac{a}{a_0} \right)^* = \frac{\lambda_a}{\lambda_{a0}} \left(\pi_0 \left(\frac{\lambda_b}{\lambda_a} \frac{\lambda_{a0}}{\lambda_{b0}} \frac{K L_{Y0}}{L_Y K_0} \right)^{\frac{\alpha\theta}{\alpha-\theta}} + 1 - \pi_0 \right)^{-\frac{1}{\alpha}}, \tag{9}$$

$$\left(\frac{b}{b_0} \right)^* = \lambda_b \left(\pi_0 + (1-\pi_0) \left(\frac{\lambda_b}{\lambda_a} \frac{\lambda_{a0}}{\lambda_{b0}} \frac{K L_{Y0}}{L_Y K_0} \right)^{-\frac{\alpha\theta}{\alpha-\theta}} \right)^{-\frac{1}{\alpha}}, \tag{10}$$

where $\frac{\alpha\theta}{\alpha-\theta}$ is substituted with $-\alpha$ in the case of Leontief LPFs ($\theta = -\infty$). Inserting these optimal technology choices into the LPF, the GPF takes the normalized CES form:

$$Y = Y_0 \left(\pi_0 \left(\lambda_b \frac{K}{K_0} \right)^{\xi} + (1-\pi_0) \left(\frac{\lambda_a}{\lambda_{a0}} \frac{L_Y}{L_{Y0}} \right)^{\xi} \right)^{\frac{1}{\xi}} \tag{11}$$

where $\xi = \frac{\alpha\theta}{\alpha-\theta} = \frac{\sigma_{GPF}-1}{\sigma_{GPF}}$, with σ_{GPF} being the aggregate or global elasticity of factor substitution. Henceforth without loss of generality we normalize $\lambda_{b0} = 1$.

Notice, comparing (11) with (3), that $\xi > \theta$, i.e., the elasticity of substitution between capital and labor is uniformly above the elasticity of substitution at the level of each technology. Thus whilst both local and global production function are CES, there is more substitutability at the aggregate level; this is certainly consistent with the idea that the global elasticity should exceed the local one, although in our case still below unity.

As mentioned, this contrasts with the Jones (2005b) model, where independent Pareto distributions of unit factor productivities were assumed, leading to aggregate Cobb-Douglas

[23]Second order conditions require us to assume that $\alpha > \theta$, so that the interior stationary point of the above optimization problem is a maximum.

[24]The final good is assumed to be the numeraire.

production. The key point is that the current setup preserves a non-unitary elasticity of substitution upon aggregation and retains factor-augmenting technical change (thus bi-directional variability in the labor share) which is one of the essential elements of the posited endogenous growth model.

4.3 R&D

We assume that new, factor-augmenting innovations are created endogenously by the respective R&D sectors (Acemoglu, 2003), augmenting the technology menu by increasing the underlying parameters λ_a, λ_b. The two R&D technologies are:

$$\dot{\lambda}_a = A\left(\lambda_b^{\phi} x^{\eta_a} \ell_a^{\nu_a}\right)\lambda_a, \tag{12}$$

$$\dot{\lambda}_b = B\left(\lambda_b^{1-\omega} x^{\eta_b} \ell_b^{\nu_b}\right) - d\lambda_b, \tag{13}$$

where ℓ_a and ℓ_b are the shares (or "research intensity") of population employed in labor- and capital-augmenting R&D, respectively, with $\ell_a + \ell_b + \ell_Y = 1$, and $\ell_Y L = L_Y$, etc. (L is total employment). Parameters A and B capture the unit productivity of the labor- and capital-augmenting R&D process, respectively. Parameter ϕ captures the spillover from capital- to labor-augmenting R&D.[25] Parameter ω, on the other hand, measures the degree of decreasing returns to scale in capital-augmenting R&D. By assuming $\omega \in (0, 1)$ we allow for the "standing on shoulders" effect in capital-augmenting R&D, albeit we limit its scope insofar as it is less than proportional to the existing technology stock (Jones, 1995). The assumption of $d > 0$ is critical for the asymptotic stability of unit capital productivity λ_b in the model, and thus for the existence of a BGP with purely labor-augmenting technical change.

The term $x \equiv \frac{\lambda_b k}{\lambda_a}$ captures the technology-corrected degree of capital-augmentation of the workplace, the "lab equipment" term. It is going to be constant along the BGP. The long-term endogenous growth engine is located in the linear labor-augmenting R&D equation. To fulfill the requirement of existence of a balanced growth path along which the growth rates of λ_a and λ_b are constant, we assume that $\eta_b\phi + \eta_a\omega \neq 0$.[26]

R&D activity, moreover, may be subject to duplication externalities (Stokey, 1995; Jones, 1995): the more numerous are researchers searching for ideas, the more likely is wasteful duplication of effort. This aspect is captured by parameters $\nu_a, \nu_b \in (0, 1]$: the higher is ν the lower the extent of duplication. This negative externality may arise from many

[25] There are no a priori restrictions on the sign of ϕ. In our baseline calibration we assume it to be positive, indicating that more efficient use of physical capital in the economy also increases the productivity of labor-augmenting R&D. See Li (2000) for a thorough discussion of the role of cross-sectoral spillovers in growth models with two R&D sectors.

[26] All our qualitative results also go through for the special case $\eta_a = \eta_b = 0$, which fully excludes "lab equipment" terms in R&D. The current inequality condition is not required in such cases.

sources, e.g., patent races and patent protection (Jones and Williams, 2000). A race to secure a lucrative medical patent, for instance, may imply large decentralized, overlapping scientific resources. Likewise, with stringent patent protection, wasteful duplication may arise since firms cannot directly build on patented technology (having to reinvent/imitate it first). On the other hand, with more patent protection, there could be less duplication because each research project gives the firm more leverage due to its patentability and exclusion of competition. The net effect is unclear.[27]

These externalities are important in our analysis. Indeed, we are not aware of any study which distinguishes between duplication externalities in labor- and capital-augmenting R&D. This raises the question of whether $\nu_a = \nu_b$, though a defensible prior, makes sense. For instance, such externalities could be stronger in labor-augmenting R&D, in so far as there is greater scope for patent protection when the technology is embodied in capital goods and subject to obsolescence.[28] Accordingly, we explore several ν_a, ν_b scenarios.

Finally, to put our forms in context, observe that equation (13) is akin to Jones' (1995) formulation, generalized by adding obsolescence and the lab equipment term. Thus, setting $d = \eta_b = 0$ retrieves Jones' original specification. And equation (12) is the same as in Romer (1990) but scale-free (it features a term in ℓ_b instead of $\ell_b \cdot L$) and with lab equipment and a direct spillover from λ_b; setting $\phi = \eta_a = 0$ retrieves the scale-free version of Romer (1990), cf. Jones (1999).

4.4 The Social Planner's Problem

The social planner maximizes the representative household's utility from discounted consumption, given standard CRRA preferences subject to the budget constraint (15) (i.e., the equation of motion of the aggregate capital stock in per-capita terms), the two R&D technologies (16)–(17), the labor-market clearing condition, (18), and the production function, (19):[29]

$$\max \int_0^\infty \frac{c^{1-\gamma} - 1}{1-\gamma} e^{-(\rho+n)t} dt \qquad s.t. \tag{14}$$

[27]In an interesting contribution, Bloom, Schankerman, and Reenen (2013) disaggregated R&D interactions across firms in "technology" space and "product" market space, finding the positive effect of the former dominating the latter negative effect.

[28]See the discussion in Solow (1960).

[29]There are three control variables, c, ℓ_a, ℓ_b and three state variables, k, λ_a, λ_b, in this optimization problem.

$$
\begin{aligned}
\dot{k} &= y - c - (\delta + n)k - \zeta \dot{a}, && (15)\\
\dot{\lambda}_a &= A\left(\lambda_b^{\phi} x^{\eta_a} \ell_a^{\nu_a}\right)\lambda_a, && (16)\\
\dot{\lambda}_b &= B\left(\lambda_b^{1-\omega} x^{\eta_b} \ell_b^{\nu_b}\right) - d\lambda_b, && (17)\\
1 &= \ell_a + \ell_b + \ell_Y, && (18)\\
y &= y_0\left(\pi_0\left(\lambda_b \frac{k}{k_0}\right)^{\xi} + (1-\pi_0)\left(\frac{\lambda_a}{\lambda_{a0}}\frac{\ell_Y}{\ell_{Y0}}\right)^{\xi}\right)^{1/\xi} && (19)
\end{aligned}
$$

where $y = Y/L$, $\gamma > 0$ is the inverse of the intertemporal elasticity of substitution, $\rho > 0$ is the rate of time preference, and $n > 0$ is the (exogenous) growth rate of the labor supply. The last term in (15) captures a negative externality that arises from implementing new labor-augmenting technologies, with $\zeta \geq 0$. Since workers need to develop skills compatible with each new technology, it is assumed that there is an external capital cost of such technology shifts (training costs, learning-by-doing, etc.).

4.5 Decentralized Allocation

Let us now proceed to a discussion of the decentralized allocation (DA) of the considered model. The construction of the decentralized allocation draws from Romer (1990), Acemoglu (2003), and Jones (2005a). In particular, we use the Dixit-Stiglitz monopolistic competition setup and the increasing variety framework of the R&D sector. The general equilibrium shall be obtained as an outcome of the interplay between: households; final goods producers; aggregators of bundles of capital- and labor-intensive intermediate goods; monopolistically competitive producers of differentiated capital- and labor-intensive intermediate goods; and competitive capital- and labor-augmenting R&D firms. We discuss these agents in turn in the following sections.

4.5.1 Households

Analogous to the social planner's optimal allocation (OA), we assume that the representative household maximizes discounted CRRA utility:

$$\max \int_0^{\infty} \frac{c^{1-\gamma} - 1}{1-\gamma} e^{-(\rho-n)t} dt \tag{20}$$

subject to the budget constraint:

$$\dot{v} = (r - \delta - n)v + w - c, \tag{21}$$

where $v = V/L$ is the household's per-capita holding of assets, $V = K + p_a\lambda_a + p_b\lambda_b$. The representative household is the owner of all capital and also holds the shares of monopolistic producers of differentiated capital- and labor-intensive intermediate goods. Capital is rented at a net market rental rate equal to the gross rental rate less depreciation: $r - \delta$. Solving the household's optimization problem yields the familiar Euler equation:

$$\hat{c} = \frac{1}{\gamma}(r - \delta - \rho), \tag{22}$$

where $\hat{c} = \dot{c}/c = g$ (g is the per-capita growth rate).

4.5.2 Final Goods Producers

The role of final goods producers is to generate the output of final goods (which are then either consumed by the representative household or saved and invested, leading to physical capital accumulation), taking bundles of capital- and labor-intensive intermediate goods as inputs. They operate in a perfectly competitive environment, where both bundles are remunerated at market rates p_K and p_L, respectively.

The final goods producers operate a normalized CES technology:

$$Y = Y_0 \left(\pi_0 \left(\frac{Y_K}{Y_{K0}} \right)^{\xi} + (1 - \pi_0) \left(\frac{Y_L}{Y_{L0}} \right)^{\xi} \right)^{\frac{1}{\xi}}. \tag{23}$$

The first order condition implies that final goods producers' demand for capital- and labor-intensive intermediate goods bundles satisfies:

$$p_K = \pi \frac{Y}{Y_K}, \; p_L = (1 - \pi) \frac{Y}{Y_L}, \tag{24}$$

where share term $\pi = \pi_0 \left(\frac{Y_K}{Y_{K0}} \frac{Y_0}{Y} \right)^{\xi}$ is the elasticity of final output with respect to Y_K.

4.5.3 Aggregators of Capital- and Labor-Intensive Intermediate Goods

There are two symmetric sectors in the economy, whose role is to aggregate the differentiated (capital- or labor-intensive) goods into the bundles Y_K and Y_L demanded by final goods producers. It is assumed that the differentiated goods are imperfectly substitutable (albeit gross substitutes). The degree of substitutability is captured by parameter $\varepsilon \in (0, 1)$:

$$Y_K = \left(\int_0^{N_K} X_{Ki}^{\varepsilon} di \right)^{\frac{1}{\varepsilon}}. \tag{25}$$

Aggregators operate in a perfectly competitive environment and decide upon their demand for intermediate goods, the price of which will be set by the respective monopolistic producers (discussed in the following subsection).

For capital-intensive bundles, the aggregators maximize:

$$\max_{X_{ki}} \left\{ p_K \left(\int_0^{N_K} X_{Ki}^{\varepsilon} di \right)^{\frac{1}{\varepsilon}} - \int_0^{N_K} p_{Ki} X_{Ki} di \right\}. \tag{26}$$

As we see, there is a continuum of measure N_K of capital-intensive intermediate goods producers. Optimization implies the following demand curve:

$$X_{Ki} = x_K(p_{Ki}) = \left(\frac{p_{Ki}}{p_K} \right)^{\frac{1}{\varepsilon-1}} Y_K^{\frac{1}{\varepsilon}}. \tag{27}$$

Symmetrically, there is also a continuum of measure N_L of labor-intensive intermediate goods producers. The demand curve for their products satisfies

$$X_{Li} = x_L(p_{Li}) = \left(\frac{p_{Li}}{p_L} \right)^{\frac{1}{\varepsilon-1}} Y_L^{\frac{1}{\varepsilon}}. \tag{28}$$

4.5.4 Producers of Differentiated Intermediate Goods

It is assumed that each of the differentiated capital- or labor-intensive intermediate goods producers, indexed by $i \in [0, N_K]$ or $i \in [0, N_L]$ respectively, has monopoly over its specific variety. It is therefore free to choose its preferred price p_{Ki} or p_{Li}. These firms operate a simple linear technology, employing either only capital or only labor.

For the case of capital-intensive intermediate goods producers, the production function is $X_{Ki} = K_i$. Capital is rented at the gross rental rate r. The optimization problem is:

$$\max_{p_{Ki}} \left(p_{Ki} X_{Ki} - r K_i \right) = \max_{p_{Ki}} (p_{Ki} - r) x_K(p_{Ki}). \tag{29}$$

The optimal solution implies $p_{Ki} = r/\varepsilon$ for all $i \in [0, N_K]$. This implies symmetry across all differentiated goods: they are sold at equal prices, thus their supply is also identical, $X_{Ki} = \bar{X}_K$ for all i. Given this regularity, market clearing implies:

$$K = \int_0^{N_K} K_i di = \int_0^{N_K} X_{Ki} di = N_K \bar{X}_K \qquad Y_K = N_K^{\frac{1-\varepsilon}{\varepsilon}} K. \tag{30}$$

The demand curve implies that the price of intermediate goods is linked to the price of the capital-intensive bundle as in $p_K = p_{Ki} N_K^{\frac{\varepsilon-1}{\varepsilon}} = \frac{r}{\varepsilon} N_K^{\frac{\varepsilon-1}{\varepsilon}}$.

Symmetrically, in the labor-intensive sector, the production function is $X_{Li} = L_{Yi}$. Employees are remunerated at the market wage rate w. The total labor supply is given by $L_Y = \ell_Y L = \int_0^{N_L} L_{Yi} di$. Optimization yields $p_{Li} = w/\varepsilon$. By symmetry, we also obtain:

$$L_Y = \int_0^{N_K} X_{Li} di = N_L \bar{X}_L \qquad Y_L = N_L^{\frac{1-\varepsilon}{\varepsilon}} L_Y. \tag{31}$$

The respective prices satisfy $p_L = p_{Li} N_L^{\frac{\varepsilon-1}{\varepsilon}} = \frac{w}{\varepsilon} N_L^{\frac{\varepsilon-1}{\varepsilon}}$.

Finally, aggregating across all the intermediate goods producers, we obtain that their total profits are equal to $\Pi_K N_K = rK\left(\frac{1-\varepsilon}{\varepsilon}\right)$ and $\Pi_L N_L = wL_Y\left(\frac{1-\varepsilon}{\varepsilon}\right)$ for capital- and labor-intensive goods respectively. Streams of profits per person in the representative household are thus $\pi_K = \Pi_K/L$ and $\pi_L = \Pi_L/L$, respectively. Hence, the total remuneration channeled to the capital-intensive sector is equal to $p_K Y_K = \frac{r}{\varepsilon}K = rK + \Pi_K N_K$, whereas the total remuneration channeled to the labor-intensive sector is equal to $p_L Y_L = \frac{w}{\varepsilon}L_Y = rL_Y + \Pi_L N_L$.

Comparing these results to the optimization problem of the final goods firms leads to,

$$r = \varepsilon\pi\frac{Y}{K} = \varepsilon\pi_0\left(\frac{Y}{K}\right)^{1-\xi}\left(\frac{Y_0}{K_0}\right)^{\xi}\left(\frac{N_K}{N_{K0}}\right)^{\xi\left(\frac{1-\varepsilon}{\varepsilon}\right)}, \tag{32}$$

$$w = \varepsilon(1-\pi)\frac{Y}{L_Y} = \varepsilon(1-\pi_0)\left(\frac{Y}{L_Y}\right)^{1-\xi}\left(\frac{Y_0}{L_{Y0}}\right)^{\xi}\left(\frac{N_L}{N_{L0}}\right)^{\xi\left(\frac{1-\varepsilon}{\varepsilon}\right)}, \tag{33}$$

$$\frac{p_K}{p_L} = \frac{\pi}{1-\pi}\frac{Y_L}{Y_K} = \frac{\pi}{1-\pi}\frac{L_Y}{K}\left(\frac{N_L}{N_K}\right)^{\frac{1-\varepsilon}{\varepsilon}} = \frac{r}{w}\left(\frac{N_L}{N_K}\right)^{\frac{1-\varepsilon}{\varepsilon}}. \tag{34}$$

In equilibrium, factor shares then amount to,

$$\pi = \pi_0\left(\frac{KY_0}{YK_0}\right)^{\xi}\left(\frac{N_K}{N_{K0}}\right)^{\xi\left(\frac{1-\varepsilon}{\varepsilon}\right)}, \tag{35}$$

$$1-\pi = (1-\pi_0)\left(\frac{L_Y Y_0}{YL_{Y0}}\right)^{\xi}\left(\frac{N_L}{N_{L0}}\right)^{\xi\left(\frac{1-\varepsilon}{\varepsilon}\right)}. \tag{36}$$

Hence, the aggregate production function, obtained after incorporating all these choices into (23), and using the definitions $\lambda_b = N_K^{\frac{1-\varepsilon}{\varepsilon}}$ and $\lambda_a = N_L^{\frac{1-\varepsilon}{\varepsilon}}$, reads:

$$Y = Y_0\left(\pi_0\left(\frac{\lambda_b K}{\lambda_{b0}K_0}\right)^{\xi} + (1-\pi_0)\left(\frac{\lambda_a L_Y}{\lambda_{a0}L_{Y0}}\right)^{\xi}\right)^{\frac{1}{\xi}}. \tag{37}$$

We see that it coincides with the aggregate production function (11) present in the social planner allocation.

4.5.5 Capital- and Labor-Augmenting R&D Firms

The role of capital- and labor-augmenting R&D firms is to produce innovations which increase the variety of available differentiated intermediate goods, either N_K or N_L. Patents never expire, and patent protection is perfect. R&D firms sell these patents to the representative household which sets up a monopoly for each new variety. Patent price, p_b or p_a, which reflects the discounted stream of future monopoly profits, is set at the competitive market. There is free entry to R&D.

R&D firms employ labor only: $L_a = \ell_a L$ and $L_b = \ell_b L$ workers are employed in the labor- and capital-augmenting R&D sectors, respectively. There is also an externality from the total physical capital stock in the economy, working through the "lab equipment" term in the R&D production function. Furthermore, the R&D firms perceive their production technology as linear in labor, while in fact it is concave due to duplication externalities.

Incorporating these assumptions and using the familiar notion $x = \frac{\lambda_b k}{\lambda_a}$, capital-augmenting R&D firms maximize:

$$\max_{\ell_b} \left(p_b \dot{N}_K - w\ell_b \right) = \max_{\ell_b} \left((p_b Q_K - w)\ell_b \right), \tag{38}$$

where $Q_K = B \left(\lambda_b^{\frac{\varepsilon}{1-\varepsilon}-\omega} x^{\eta_b} \ell_b^{\nu_b - 1} \right) \left(\frac{\varepsilon}{1-\varepsilon} \right)$ is treated by firms as a constant in the steady state (Romer, 1990; Jones, 2005a). Analogously, labor-augmenting R&D firms maximize:

$$\max_{\ell_a} \left(p_a \dot{N}_L - w\ell_a \right) = \max_{\ell_a} \left((p_a Q_L - w)\ell_a \right), \tag{39}$$

where $Q_L = A \left(\lambda_a^{\frac{\varepsilon}{1-\varepsilon}} \lambda_b^{\phi} x^{\eta_a} \ell_a^{\nu_a - 1} \right) \left(\frac{\varepsilon}{1-\varepsilon} \right)$ is treated as exogenous.

Free entry into both R&D sectors implies $w = p_b Q_K = p_a Q_L$. Purchase of a patent entitles the holders to a per-capita stream of profits equal to π_K and π_L, respectively. While the production of any labor-augmenting varieties lasts forever, there is a constant rate d at which production of capital-intensive varieties becomes *obsolete*. This effect is external to patent holders and thus is not strategically taken into account when accumulating the patent stock.

4.5.6 Equilibrium

We define the *decentralized equilibrium* as the collection of time paths of all the respective quantities: $c, \ell_a, \ell_b, k, \lambda_b, \lambda_a, Y_K, Y_L, \{X_{Ki}\}, \{X_{Li}\}$ and prices $r, w, p_K, p_L, \{p_{Ki}\}, \{p_{Li}\}, p_a, p_b$ such that: (1) households maximize discounted utility subject to their budget constraint; (2) profit maximization is followed by final-goods producers, aggregators and producers of capital- and labor-intensive intermediate goods, and capital- and labor-augmenting R&D firms; (3) the labor market clears: $L_a + L_b + L_Y = (\ell_a + \ell_b + \ell_Y)L = L$; (4) the asset market clears: $V = vL = K + p_a\lambda_a + p_b\lambda_b$, where assets have equal returns: $r - \delta = \frac{\pi_L}{p_a} + \frac{\dot{p}_a}{p_a} = \frac{\pi_K}{p_b} + \frac{\dot{p}_b}{p_b} - d$; and, finally (5), such that the aggregate capital stock satisfies $\dot{K} = Y - C - \delta K - \zeta \dot{a} L$, where the last term is an externality term (as previously discussed).

4.6 Solving for the Social Planner Allocation

4.6.1 Balanced Growth Path

Since Uzawa (1961) we have known that any neoclassical growth model – including the one laid out above – can exhibit balanced growth if technical change is purely labor-augmenting or if production is Cobb-Douglas. Hence, once we presume a CES production function, the analysis of dynamic consequences of technical change, which is not purely labor-augmenting, must be done outside the BGP.

Along the BGP, we obtain the following growth rate of key model variables:

$$g = \hat{\lambda}_a = \hat{k} = \hat{c} = \hat{y} = A(\lambda_b^*)^{\phi} (x^*)^{\eta_a} (\ell_a^*)^{\nu_a}, \tag{40}$$

where stars denote steady-state values.

Ultimately long-run growth is driven by labor-augmenting R&D. This can be explained by the fact that labor is the only non-accumulable factor, and that it is complementary to capital along the aggregate production function. The following variables are constant along the BGP: $y/k, c/k, \ell_a, \ell_b$ and λ_b (thus asymptotically there is no capital-augmenting technical change).

4.6.2 Externality Term

The externality term in the social planner's optimization problem can be computed using the firms' optimal technology choice. In general, we derive:

$$a = \lambda_a (N(1-\pi_0))^{1/\alpha} \left(\frac{\pi}{\pi_0}\right)^{1/\alpha} \quad \forall t. \tag{41}$$

Hence the formula for $\dot{a}$ becomes rather involved. To make the externality more tractable, we assume that,

$$\frac{\dot{a}}{k} \propto g\left(\frac{\lambda_b}{x}\left(\frac{\pi}{\pi_0}\right)^{1/\alpha}\right) \tag{42}$$

which is an identity at the BGP and a first-order approximation outside of it.

4.6.3 Euler Equations

Having set up the Hamiltonian and computed its derivatives, the following Euler equations are obtained for the social planner allocation:

$$\hat{c} = \frac{1}{\gamma}\left(\frac{y}{k}\left(\pi + \frac{1-\pi}{\ell_Y}\left(\frac{\eta_a \ell_a}{\nu_a} + \frac{\eta_b \ell_b}{\nu_b}\right)\right) - \delta - \rho\right), \tag{43}$$

$$\varphi_1 \hat{\ell}_a + \varphi_2 \hat{\ell}_b = Q_1, \tag{44}$$

$$\varphi_3 \hat{\ell}_a + \varphi_4 \hat{\ell}_b = Q_2, \tag{45}$$

where

$$\varphi_1 = \nu_a - 1 - (1-\xi)\pi\frac{\ell_a}{\ell_Y}, \quad (46)$$

$$\varphi_2 = -(1-\xi)\pi\frac{\ell_b}{\ell_Y} \quad (47)$$

$$\varphi_3 = -(1-\xi)\pi\frac{\ell_a}{\ell_Y} \quad (48)$$

$$\varphi_4 = \nu_b - 1 - (1-\xi)\pi\frac{\ell_b}{\ell_Y}, \quad (49)$$

$$Q_1 = -\gamma\hat{c} - \rho + n + \hat{\lambda}_a\left(\frac{\ell_Y\nu_a}{\ell_a} + 1 - \eta_a - \eta_b\frac{\ell_b\nu_a}{\ell_a\nu_b}\right) - \phi\hat{\lambda}_b + ((1-\xi)\pi - \eta_a)\hat{x} \quad (50)$$

$$\begin{aligned} Q_2 &= -\gamma\hat{c} - \rho + n + \hat{\lambda}_a + \hat{\lambda}_b\left(\frac{\pi}{1-\pi}\frac{\ell_Y\nu_b}{\ell_b} + (\phi+\eta_a)\frac{\nu_b\ell_a}{\nu_a\ell_b} + \eta_b\right) \\ &+ ((1-\xi)\pi - \eta_b)\hat{x} + d\left(\frac{\pi}{1-\pi}\frac{\ell_Y\nu_b}{\ell_b} + (\phi+\eta_a)\frac{\nu_b\ell_a}{\nu_a\ell_b} - \omega + \eta_b\right). \end{aligned} \quad (51)$$

A sufficient condition for all transversality conditions to be satisfied is that $(1-\gamma)g + n < \rho$.

4.6.4 Steady State of the Transformed System

The steady state of the transformed dynamical system implied by the social planner solution (i.e., in coordinates: $u = (c/k), \ell_a, \ell_b, x, \lambda_a$, with auxiliary variables $z = (y/k), \pi, g$) satisfies:

$$g = \hat{\lambda}_a = \hat{k} = \hat{c} = \hat{y} = A(\lambda_b^*)^{\phi}\,(x^*)^{\eta_a}\,(\ell_a^*)^{\nu_a} \quad (52)$$

$$\gamma g + \delta + \rho = z\left(\pi + \frac{1-\pi}{\ell_Y}\left(\frac{\eta_a\ell_a}{\nu_a} + \frac{\eta_b\ell_b}{\nu_b}\right)\right) \quad (53)$$

$$g = z - \zeta\frac{\dot{a}}{k} - u - (\delta + n) \quad (54)$$

$$d = B\left(\lambda_b^{-\omega}x^{\eta_b}\ell_b^{\nu_b}\right) \quad (55)$$

$$(1-\gamma)g + n - \rho = d\left(\frac{\pi}{1-\pi}\frac{\ell_Y\nu_b}{\ell_b} + (\phi+\eta_a)\frac{\nu_b\ell_a}{\nu_a\ell_b} - \omega + \eta_b\right) \quad (56)$$

$$(1-\gamma)g + n - \rho = -g\left(\frac{\ell_Y\nu_a}{\ell_a} - \eta_a - \eta_b\frac{\ell_b\nu_a}{\ell_a\nu_b}\right) \quad (57)$$

$$\frac{\pi}{\pi_0} = \left(\frac{\lambda_b}{\lambda_{b0}}\right)^{\xi}\left(\frac{z}{z_0}\right)^{-\xi} \quad (58)$$

$$\frac{z}{z_0} = \frac{\lambda_b}{\lambda_{b0}}\left(\pi_0 + (1-\pi_0)\left(\frac{x_0}{x}\frac{\ell_Y}{\ell_{Y0}}\right)^{\xi}\right)^{1/\xi}. \quad (59)$$

This non-linear system of equations will be solved numerically, yielding the unique steady state of the de-trended system, and thus the unique BGP of the model in original variables. All further analysis of the social planner allocation will be based on the (numerical) linearization of the 5-dimensional dynamical system of equations (43)–(45), (15) and (17), taking the BGP equality (40) as given.

4.7 Solving for the Decentralized Allocation

When solving for the decentralized allocation, we broadly follow the steps carried out in the case of the social planner allocation. We first solve analytically for the BGP of our endogenous growth model and then linearize the implied dynamical system around the BGP.

4.7.1 Balanced Growth Path

Along the BGP, we obtain the following growth rate of the key model variables:

$$g = \hat{k} = \hat{c} = \hat{y} = \hat{w} = \hat{p}_b = \hat{p}_{Li} = \hat{\lambda}_a = A(\lambda_b^*)^{\phi}\,(x^*)^{\eta_a}\,(\ell_a^*)^{\nu_a}. \tag{60}$$

The following quantities are constant along the BGP: $y/k, c/k, \ell_a, \ell_b, Y_K/Y, Y_L/Y$ and λ_b (again, asymptotically, no capital-augmenting technical change). The following prices are also constant along the BGP: $r, p_a, p_K, p_L, \{p_{Ki}\}$.

4.7.2 Euler Equations

Calculations imply that the decentralized equilibrium is associated with the following Euler equations describing the first-order conditions:

$$\hat{c} = \frac{1}{\gamma}\left(\varepsilon\pi\frac{y}{k} - \delta - \rho\right), \tag{43'}$$

$$\varphi_1\hat{\ell}_a + \varphi_2\hat{\ell}_b = \tilde{Q}_1, \tag{44'}$$

$$\varphi_3\hat{\ell}_a + \varphi_4\hat{\ell}_b = \tilde{Q}_2, \tag{45'}$$

where

$$\tilde{Q}_1 = -\varepsilon\pi\frac{y}{k} + \delta + \hat{\lambda}_a\frac{\ell_Y}{\ell_a} - \phi\hat{\lambda}_b + ((1-\xi)\pi - \eta_a)\hat{x} \tag{50'}$$

$$\tilde{Q}_2 = -\varepsilon\pi\frac{y}{k} + \delta + \hat{\lambda}_a + (\hat{\lambda}_b + d)\left(\frac{\pi}{1-\pi}\frac{\ell_Y}{\ell_b}\right) - \hat{\lambda}_b(1-\omega) - d + ((1-\xi)\pi - \eta_b)\hat{x} \tag{51'}$$

and where φ_1 through φ_4 are defined as in equations (46)–(49). A sufficient condition for all transversality conditions to be satisfied is that $(1-\gamma)g + n < \rho$.

4.7.3 Departures from the Social Optimum

Departures of the decentralized allocation from the optimal one can be tracked back to some specific assumptions regarding the information structure of the decentralized allocation. Those differences are the following:

1. In the consumption Euler equation, comparing equations (43) with (43′), the term $\frac{y}{k}\left(\pi + \frac{1-\pi}{\ell_Y}\left(\frac{\eta_a \ell_a}{\nu_a} + \frac{\eta_b \ell_b}{\nu_b}\right)\right)$ is replaced by $\varepsilon\pi\frac{y}{k}$. This is due to two effects:

   (a) in contrast to the social planner, markets fail to account for the external effects of physical capital on R&D activity via the lab equipment terms (with respective elasticities η_b and η_a);

   (b) ε appears in the decentralized allocation due to imperfect competition in the labor- and capital-augmenting intermediate goods sectors.

2. In the Euler equations for ℓ_a and ℓ_b (equations (44), (45), (44′), (45′)) the shadow price of physical capital $\hat{c} - \rho + n$ is replaced by its market price $r - \delta = \varepsilon\pi\frac{y}{k} - \delta$ which accounts for markups arising from imperfect competition.

3. In the Euler equation for ℓ_a, the term $\left(\frac{\ell_Y \nu_a}{\ell_a} + 1 - \eta_a - \eta_b \frac{\ell_b \nu_a}{\ell_a \nu_b}\right)$ is replaced by $\frac{\ell_Y}{\ell_a}$. This is due to two effects:

   (a) ν_a is missing because markets fail to internalize the labor-augmenting R&D duplication effects when $\nu_a < 1$;

   (b) the latter two components are missing because markets fail to account for the external effects of accumulating knowledge on future R&D productivity. These effects are included in the shadow prices of λ_a and λ_b in the social planner allocation but not in their respective market prices.

4. Analogously, in the Euler equation for ℓ_b, the term $\left(\frac{\ell_Y \nu_b}{\ell_b}\frac{\pi}{1-\pi} + (1-\omega) + \eta_b + (\phi + \eta_a)\frac{\ell_a \nu_b}{\ell_b \nu_a}\right)$ is replaced by $\left(\frac{\ell_Y}{\ell_b}\frac{\pi}{1-\pi}\right)$. The same reasoning follows as per point 3.

4.7.4 Steady State of the Transformed System: Decentralized Solution

The steady state of the transformed system satisfies:

$$g = \hat{\lambda}_a = \hat{k} = \hat{c} = \hat{y} = A(\lambda_b^*)^{\phi} (x^*)^{\eta_a} (\ell_a^*)^{\nu_a} \tag{61}$$

$$\gamma g + \rho = r - \delta \tag{62}$$

$$g = z - \zeta \frac{\dot{a}}{k} - u - (\delta + n) \tag{63}$$

$$d = B \left(\lambda_b^{-\omega} x^{\eta_b} \ell_b^{\nu_b} \right) \tag{64}$$

$$g \frac{\ell_Y}{\ell_a} = r - \delta \tag{65}$$

$$g = r - \delta + d \left(1 - \frac{\pi}{1-\pi} \frac{\ell_Y}{\ell_b} \right) \tag{66}$$

$$r = \varepsilon \pi z \tag{67}$$

$$\frac{\pi}{\pi_0} = \left(\frac{\lambda_b}{\lambda_{b0}} \right)^{\xi} \left(\frac{z}{z_0} \right)^{-\xi} \tag{68}$$

$$\frac{z}{z_0} = \frac{\lambda_b}{\lambda_{b0}} \left(\pi_0 + (1-\pi_0) \left(\frac{x_0}{x} \frac{\ell_Y}{\ell_{Y0}} \right)^{\xi} \right)^{1/\xi} . \tag{69}$$

This non-linear system of equations will be solved numerically, yielding the unique steady state of the de-trended system, and thus the unique BGP of the model in original variables. All further analysis of the decentralized allocation will be based on the (numerical) linearization of the 5-dimensional dynamical system of equations (43′)–(45′), (15) and (17), taking the BGP equality (60) as given.

5 Model Solution and Dynamics

5.1 Calibration

The model calibration for the decentralized system is listed in **Table 7**. For most parameters we use values typically found in the literature, or straightforwardly retrieved from the data. Others are more difficult to calibrate. This is because of the scarcity of some key data and, more generally, the imperfect mapping between model concepts and observables. To counter this uncertainty, we provide an in-depth robustness check by varying key parameters over a plausible support.

The calibration itself follows five main steps. First, several "deep" parameters are pre-determined by taking values stemming from the associated literature: namely, the (inverse) intertemporal elasticity of substitution, and the rate of time preference.

Second, we assign CES normalization parameters to match US long-run averages, spanning 1929–2012 for factor income shares, and post-war estimates for the aggregate substitution elasticity.[30]

Third, we assume that a range of long-run averages from US data correspond to the BGP of (the decentralized allocation of) the model. Doing so allows us to calibrate the following variables: the rates of economic and population growth, capital productivity, and the consumption-to-capital ratio.

Next, with this in hand, four identities included in the system (61)–(69) drive the calibration of other parameters in a model-consistent manner: $\ell_y^*, r^*, \lambda_b^*, x^*$ and ε. Final production employment is set in a model-consistent manner. In the absence of any other information, we agnostically assume that the share $1 - \ell_Y^*$ is split equally between employment in both R&D sectors.

For the model-consistent value of ℓ_Y^*, this formula leads to relatively high research employment shares. But that may be defended on a number of grounds. First, the model's limited occupational granularity mechanically produces such an outcome (i.e., employment is either in R&D or in final-goods production).[31] Accordingly, layering additional non R&D-associated sectors would automatically reduce these shares, without changing the underlying mechanisms of interest in the model.

However beyond that, counting the number of scientists, researchers, teachers and even patents and expenditures has long been recognised as a crude proxy for research activity.[32] In this respect, we might instead choose to interpret the ℓ_a^* and ℓ_b^* values as a correction for the managerial and entrepreneurial input to production as well as learning-by-doing on the side of employees; when new technologies are implemented in production, they require significant effort and/or reorganization of the workplace, which might be considered to show up as R&D in our simplified model. Similarly, it may capture non-routine and analytical tasks in the employment spectrum which do not necessarily show up in formal research-intensive job definitions (see Autor, Levy, and Murnane, 2003).

[30] Although, Klump, McAdam, and Willman (2012) found that even studies based on a very long period of history reported estimates of the US elasticity of substitution below one. For instance, the seminal Arrow, Chenery, Minhas, and Solow (1961) paper found an aggregate elasticity over 1909-1949, of 0.57.

[31] It may though be moved up or down to some degree depending on what one assumes for preferences: e.g., as agents become more impatient, less labor is allocated to R&D: $\partial \ell_Y^* / \partial \rho > 0$. The same holds for the inter-temporal substitution elasticity.

[32] See the "Oslo Manual" (OECD/Eurostat, 2005) for a discussion of the various R&D types, and measurement issues.

Table 7: Baseline Calibration: Pre-Determined Parameters (Decentralized Allocation)

Parameter		Value	Source/Target
Preferences			
Inverse Intertemporal Elasticity of Substitution	γ	1.7500	Barro and Sala-i-Martin (2003)
Time Preference	ρ	0.0200	Barro and Sala-i-Martin (2003)
Income and Production			
GDP Per-Capita Growth	g	0.0171	geometric average of gross growth rates
Population Growth rate	n	0.0153	geometric average of gross growth rates
Labor in Aggregate Production	ℓ_{Y0}, ℓ_Y^*	0.5394	$\frac{0.5\left(\gamma+\frac{\rho}{g}\right)}{1+0.5\left(\gamma+\frac{\rho}{g}\right)}$
Capital Productivity	z_0, z^*	0.3442	geometric average
Consumption-to-Capital	u^*	0.2199	geometric average
Capital Income Share	π_0, π^*	0.3260	arithmetic average
Depreciation	δ	0.0600	Caselli (2005)
Factor Substitution Parameter[†]	θ	−0.7500	$\sigma = 0.7$, Klump, McAdam, and Willman (2012)
Net Real Rate of Return	$r^* - \delta$	0.0499	$r^* - \delta = \gamma g + \rho$
Substitutability Between Intermediate Goods	ε	0.9793	$\varepsilon = \frac{r^*}{\pi^* z^*}$
R&D Sectors			
R&D Duplication Parameters	$\nu_a = \nu_b$	0.7500	see text
Technology-Augmenting Terms	$\lambda_{a0}, \lambda_{b0}$	1.0000	see text
Technology-Augmenting Terms	λ_b^*	1.0000	$\lambda_b^* = \lambda_{b0}\frac{z^*}{z_0}\left(\frac{\pi^*}{\pi_0}\right)^{\frac{1}{\xi}}$
Labor Input in R&D sectors	ℓ_Y^a, ℓ_Y^b	0.2033	$\ell_a^* = \ell_b^*$ for $\ell_a^* + \ell_b^* = 1 - \ell_Y^*$
Lab-Equipment Term[‡]	x_0, x^*	61.7900	$x^* = x_0\frac{\ell_Y^*}{\ell_{Y0}}\left(\frac{1}{1-\pi_0}\left(\frac{z^*}{z_0}\frac{\lambda_{b0}}{\lambda_b^*}\right)^{\xi} - \frac{\pi_0}{1-\pi_0}\right)^{-1/\xi}$

Notes: † We can plausibly set $\alpha = 1$ and determine the aggregate elasticity of substitution via θ; $\xi = \frac{\alpha\theta}{\alpha-\theta} = -0.43$. This is consistent with $\xi = \frac{\sigma-1}{\sigma}$, where the aggregate elasticity of factor substitution $\sigma = 0.7$. ‡ $x_0 = \frac{\lambda_{b0}k_0}{\lambda_{a0}} = 61.79$.

Table 8: Baseline Calibration: Free Parameters

Parameter		Value
Labor-Augmenting R&D		
Unit productivity	A	0.0210
Lab equipment exponent	η_a	0.2404
Capital-Augmenting R&D		
Unit productivity	B	0.1565
Lab equipment exponent	η_b	0.1256
Degree of decreasing returns	ω	0.5000
Spillover from capital- to labor-augmenting tech. change	ϕ	0.3000
Obsolescence rate	d	0.0795
Technology choice externality	ζ	115.2793

As regards the duplication externalities in factor-augmenting R&D, the literature is generally agnostic about their magnitude (Jones, 2002). Moreover, as earlier stated, the literature typically considers a *unique* R&D duplication externality. We agnostically set $\nu_a = \nu_b$, using the value in Jones and Williams (2000).[33] The technology-augmenting term λ_b^* is set in a model-consistent manner, and following Section 4.2, we set its normalized value to unity.

The final step is to assign values to the remaining parameters, in particular the technological parameters of R&D equations. We do this by solving the four remaining equations in the system (61)–(69) with respect to the remaining parameters, see **Table 8**.[34] Given this benchmark calibration, the steady state is a saddle point.

Having assessed this baseline calibration, we also examine perturbations of key parameters as a robustness exercise. First, we consider whether there are differences in equilibrium labor share between the decentralized and the optimal allocation.

5.2 BGP Comparisons

5.2.1 Is the Decentralized Labor Share Socially Optimal?

A familiar outcome in this class of models is that the socially-optimal growth rate exceeds the decentralized one: this is due to the absence of monopolistic markups plus the social value of innovations captured by the external effects of capital in R&D. And this is what we see when we solve the model (**Table 9**): the BGP of the decentralized solution features lower growth but higher consumption (u is higher). With lower growth, capital costs

[33]This can also be justified as being an average of the original constant-returns-to-scale parametrization of $\nu = 1$ (Romer, 1990) and the evidence of $\nu \approx 0.5$ provided by Pessoa (2005).

[34]In his estimations, Pessoa (2005) uses values for the obsolescence parameter between zero and fifteen percent; our endogenously-determined value is thus in the middle of that range.

are cheaper and the net real rate of return of capital is higher, and capital productivity is accordingly higher.

Table 9: BGP Comparison under the Baseline Calibration

Variable		DA	OA
Output growth rate	g	0.0171	0.0339
Consumption-to-capital ratio	u^*	0.2199	0.1628
Capital productivity	z^*	0.3442	0.3071
Employment in production	ℓ_Y^*	0.5934	0.4385
Share of labor-augmenting R&D	ℓ_a^*	0.2033	0.2575
Share of capital-augmenting R&D	ℓ_b^*	0.2033	0.3040
Capital income share	π^*	0.3261	0.2146
Labor income share	$1-\pi^*$	0.6739	0.7854
Net real rate of return	$r^*-\delta$	0.0499	0.0059
Capital-augmenting technology	λ_b^*	1.0000	2.3696
Lab equipment	x^*	61.7900	173.3363

That there should be a growth differential in favor of the social planner is straightforward. It follows from our discussion in Section 4.7.3. But the comparison in terms of the labor share is perhaps less obvious. In fact we see the striking result that the labor share in the decentralized equilibrium is 11 pp. *below* the social optimum. A series of checks (see **Figure 4**) confirms that such a large discrepancy is highly robust, see the following section.

To understand why, let us decompose the capital income share, π, in the following two ways (recalling that here the labor income share is $1-\pi$):

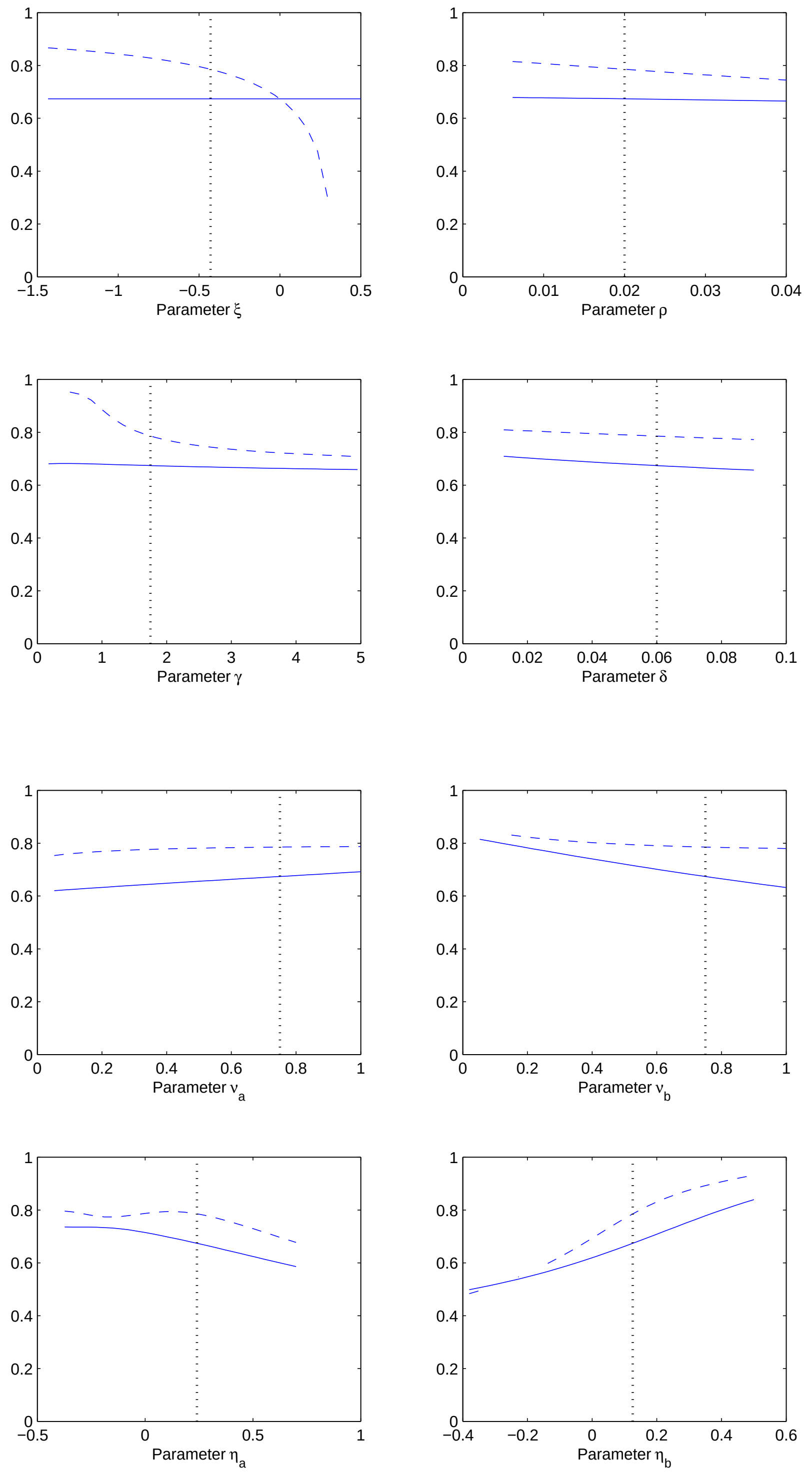
$$\frac{\pi}{\pi_0} = \left(\frac{\lambda_b k}{k_0}\right)^{\xi}\left(\frac{y}{y_0}\right)^{-\xi} \quad \Rightarrow \quad \hat{\pi} = \xi(\hat{\lambda}_b + \hat{k} - \hat{y}), \tag{70}$$

$$\frac{\pi}{1-\pi} = \frac{\pi_0}{1-\pi_0}\left(\frac{x}{x_0}\frac{\ell_{Y0}}{\ell_Y}\right)^{\xi} \quad \Rightarrow \quad \hat{\pi} = \xi(1-\pi)(\hat{x} - \hat{\ell}_Y). \tag{71}$$

Equation (70) shows that under gross complementarity ($\xi < 0$), the capital share decreases with inverse capital productivity and with capital augmentation (i.e, the capital-augmenting technology improvements are "labor biased").

Equation (71), in turn, follows from the definition of the aggregate production function and the "lab equipment" term x. Given $\hat{\ell}_Y \equiv -\left(\frac{\ell_a}{\ell_Y}\hat{\ell}_a + \frac{\ell_b}{\ell_Y}\hat{\ell}_b\right)$, the dynamics of employment in the goods sector are equal to the inverse of the dynamics of total R&D employment. It then follows that dynamics of the labor share are *uniquely determined* by the sum of the dynamics of the lab equipment component and R&D employment. Likewise, the sign of this relationship depends upon the substitution elasticity: if $\xi < 0$ then increases in R&D intensity reduce π, and thus increase the labor share, and vice versa.

Figure 4: Dependence of the Equilibrium Labor Share on the Model Parametrization



Notes: $1 - \pi$ on vertical axis; corresponding parameter support on the horizontal axis. Social planner allocation (dashed lines), decentralized equilibrium (solid lines). The vertical dotted lines in each graph represents the baseline calibrated parameter values.

Comparing the decentralized and the social planner's allocation through the lens of equation (70), we observe that the large difference in factor shares at the BGP is driven almost exclusively by the difference in the level of capital augmentation λ_b^*. This result underscores our initial hypothesis in the Introduction that technical change is quantitatively more important for explaining the labor share than the share of the capital stock in output.

Equivalently, by equation (71), this large difference in the degree of capital augmentation shows up in the lab equipment term x^*. It is also strengthened by the discrepancy in employment in final production ℓ_Y^*, which is higher in the decentralized allocation because the planner devotes more resources to (both types of) R&D. Thanks to this, coupled with relatively more saving, it achieves faster growth at the BGP but with a lower consumption-to-capital ratio and a lower rate of return to capital. All of these make for a higher labor share in the optimal allocation.

5.2.2 Impact of Parameter Variation on Labor Share at the BGP

In the light of equations (70) and (71), the panels in **Figure 4** make sense. As agents become less patient, R&D intensity falls, as does the labor share. Similar reasoning pertains to the inverse elasticity of substitution. That $\frac{\partial(1-\pi)}{\partial \eta_b} > 0$ arises from the usual property that, under gross complements, improvements in capital-augmenting technical change are labor biased; equivalent reasoning pertains to $\frac{\partial(1-\pi)}{\partial \eta_a} < 0$. Likewise, given that ℓ_a and ℓ_b are in the unit interval, we have under gross complements: $\frac{\partial(1-\pi)}{\partial \nu_a} > 0, \frac{\partial(1-\pi)}{\partial \nu_b} < 0$. If capital depreciates faster, the capital (labor) share rises (falls).

The figure also reveals that the only case where the decentralized allocation leads to a relatively higher labor share is when capital and labor are gross substitutes ($\xi > 0$). Note that the lack of dependence of the BGP on ξ in the decentralized allocation follows from CES normalization (Klump and de La Grandville, 2000), coupled with the fact that we have calibrated the normalization constants to the BGP of the decentralized allocation. A more extensive study of the dependence of both BGPs on key model parameters $(\xi, \rho, \gamma, \nu_b)$ is included in the Appendix.

5.3 Dynamics of the Labor Share

Given our baseline calibration, both allocations exhibit endogenous, dampened oscillations of the labor share and other de-trended model variables, see **Table 10**. The decentralized allocation features relatively shorter cycles but also faster convergence to the BGP. Hence, it cannot be claimed directly that the decentralized equilibrium has *excessive volatility*. If both allocations were to start from the same initial point outside of the BGP then the decentralized allocation would exhibit a greater frequency but smaller amplitude of cyclical variation.

Table 10: Dynamics Around the BGP under the Baseline Calibration

Allocation	DA	OA
Pace of convergence* (% per year)	6.3%	4.2%
Length of full cycle† (years), **L**	52.6	76.7

Note: * computed as $1 - e^{rr}$ where $rr < 0$ is the real part of the largest stable root; † computed as $\mathbf{L} = 2\pi/ir$ where $ir > 0$ is the imaginary part of two conjugate stable roots (if they exist).

Having scrutinized the robustness of this dynamic result by extensively altering the model parametrization, we conclude that while the decentralized equilibrium generally exhibits shorter cycles, the ordering of both allocations in terms of the pace of convergence can sometimes be reversed. This finding lends partial support to the claim that the decentralized equilibrium is perhaps likely to feature greater labor share volatility compared to the social optimum. We also note that as opposed to claimed discussed in some literature (e.g. Piketty, 2014), oscillations in the labor income share can be socially optimal in this model.

5.4 Emergence of Stable Limit Cycles via Hopf Bifurcations

Let us now turn to the question if one could plausibly expect limit cycle behavior of the labor share. At the baseline calibration, the answer is negative because the model exhibits oscillatory convergence to the BGP. On the other hand, we know that by Hopf's bifurcation theorem (see e.g. Feichtinger, 1992), if when exploring the support of one of the model parameters, real parts of two stable conjugate roots of the system transversally cross zero, the steady state loses its stability and a stable limit cycle is created around it.[35] The question then is does such a situation appear in our case, and if so, does it occur around an empirically plausible parameter set?

In addition to Hopf bifurcations, another interesting type of sudden changes in model dynamics can be observed here: a "node-focus" bifurcation. If, manipulating one of the model parameters, two stable real eigenvalues collide and become complex conjugates, then the pattern of convergence to the steady state changes from monotonic to oscillatory.

Accordingly, we carried out the following numerical exercise. We computed the eigenvalues of the system around the steady state of the model for various values of one certain parameter in question, assuring that whatever assumption on its value is made, other "free" parameters are re-calibrated in a way that the model always remains in perfect accordance with BGP characteristics. In this way, we are able to identify model parametriza-

[35] More precisely, in the multi-dimensional case with which we are dealing here, at the point of a Hopf bifurcation the steady state ceases to be a stable focus along the stable manifold, and becomes a repelling focus instead, and a stable limit cycle around the steady state is created in the subspace formerly referred to as the stable manifold.

tions leading to various types of local dynamics around *the same* BGP.

Figures 5–6 illustrate the results of such "multi-calibration" exercise (or "BGP-preserving" sensitivity analysis) for ρ (time preference), γ (inverse elasticity of intertemporal substitution), $\frac{\ell_a^*}{\ell_a^*+\ell_b^*}$ (the share of labor-augmenting R&D at the BGP) as well as ξ (and thus the elasticity of substitution) and η_b (the lab equipment term in capital-augmenting R&D).[36] *Hopf bifurcations*, where a stable limit cycle around the steady state is created, are obtained when manipulating the three former parameters: ρ, γ and $\frac{\ell_a^*}{\ell_a^*+\ell_b^*}$. Specifically, real parts of a pair of conjugate eigenvalues cross zero when ρ and γ become sufficiently small (below 0.0062 and 0.9430, respectively). When the steady-state ratio of labor-augmenting R&D to total R&D employment is sufficiently small (around 0.41), a stable limit cycle is created.

Such values are in the ballpark of empirically plausible ones. To get a limit cycle in factor shares (as well as other model parameters), it suffices that the household is just slightly more willing to substitute consumption across time than under logarithmic preferences. Although the critical value of the time preference rate is rather low in the baseline case, it should be noted that with $\gamma = 1$ (log preferences), the bifurcation value with respect to ρ appears already around 0.019, very close to its baseline value.

"Node–focus" bifurcations, on the other hand, are found when manipulating ξ (the factor substitutability parameter), η_b (the "lab equipment" term in capital-augmenting technical change), and again $\frac{\ell_a^*}{\ell_a^*+\ell_b^*}$. Specifically, we observe that when ξ becomes sufficiently large (and already above zero so that we are in the range of gross factor substitutability), then the imaginary parts of two conjugate stable roots hit zero, so that oscillatory dynamics are eliminated. Under gross complementarity, the magnitude of input complementarity generally increases the frequency of observed oscillations.

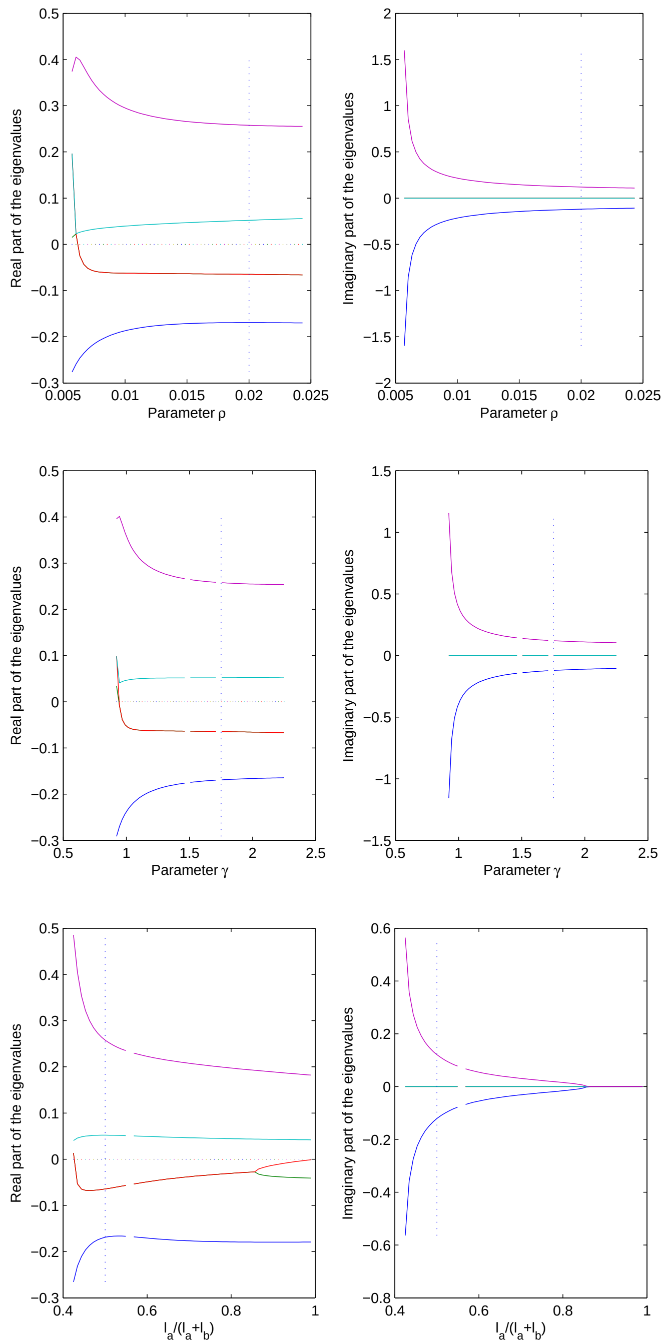
Another finding is that η_b, measuring the impact of lab equipment on the productivity of capital-augmenting R&D, is also conducive to "node–focus" bifurcations. Oscillatory dynamics prevail only if η_b is sufficiently low, and the lower it is, the higher the oscillation frequency. By the same token, when the steady-state share of labor-augmenting R&D in total R&D employment is sufficiently high (above 90%), dampened oscillations disappear in favor of monotonic convergence.

Finally, as far as manipulations in ξ are concerned, we also observe a further phenomenon. Namely, in the range of gross substitutability there exists a critical value of ξ when a real root switches its sign. At this point a *generalized saddle-node bifurcation* appears, due to which the dynamics around the steady state switch from locally indeterminate to fully determinate. If ξ is above a specific threshold value then there exists a unique saddle path, along which convergence to the steady state is monotonic.

Moreover, the implied eigenvectors can be used for inferring our theoretical predictions on the cyclical co-movement of the original model variables (including the labor share $1-\pi$). It is predicted, both for the decentralized and optimal allocation, that all

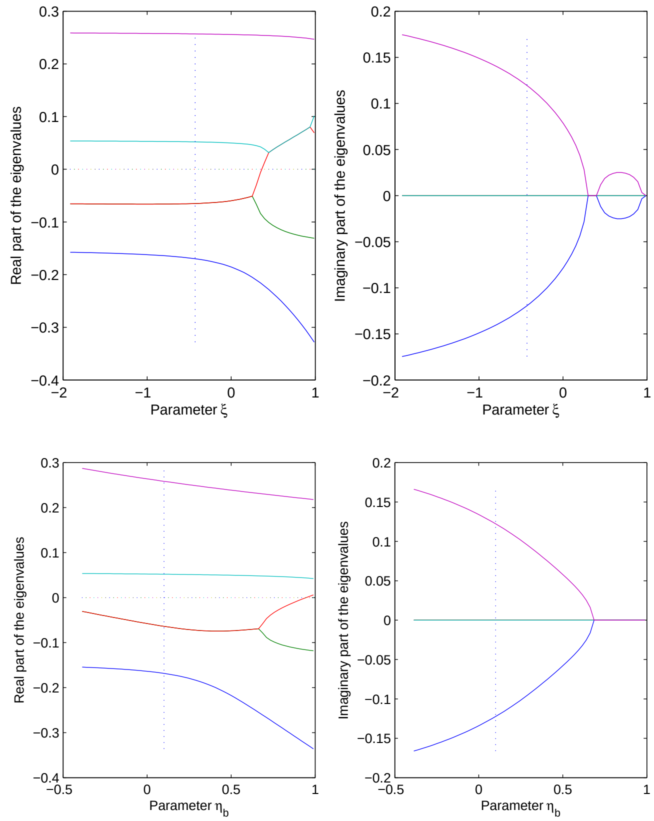
[36]Results for the other parameters – δ, η_a, ν_b, and ν_a – have been relegated to the appendix as they do not produce qualitative changes in model dynamics.

Figure 5: Emergence of Limit Cycles via Hopf Bifurcations



variables except for the consumption-capital ratio $u = c/k$ oscillate when converging to the steady state, with the same frequency of oscillations. The level of capital-augmenting technology λ_b, the "lab equipment" term x, and labor-augmenting R&D employment ℓ_a are always pro-cyclical (i.e., are positively functionally related to the economic growth rate g), whereas the cyclicality of capital-augmenting R&D ℓ_b is ambiguous (in the baseline calibration, ℓ_b is countercyclical in the decentralized allocation but pro-cyclical in op-

Figure 6: "Node–focus" Bifurcations



timal one). Furthermore, as long as capital and labor are gross complements ($\xi < 0$), the labor income share $1 - \pi$ is unambiguously pro-cyclical as well. These features of cyclical co-movement align well with the empirical evidence for the US medium-term cycle. In particular, the US labor share is indeed procyclical over the medium run – despite its countercyclicality along the business cycle.

Finally, as demonstrated in the Appendix **(Figures C.7–C.8)**, the pattern of dependence of model dynamics on its parameters is generally quite similar in the case of the decentralized equilibrium and the social planner allocation; differences are only quantitative. This suggests that emergence of stable limit cycles via Hopf bifurcations as well as "node-focus" bifurcations can readily occur in both cases.[37]

[37]Note that the bifurcation graphs in **Figures C.7–C.8** are non-BGP-preserving. Unlike **Figures 5–6**, the model has not been sequentially re-calibrated here for to keep the BGP intact. The reason for this change in approach is that a different re-calibration would be necessary for each of the two allocations, and that would impede their comparability.

5.5 Labor Share Cycles and Duplication Externalities

In the following exercises, we isolate the effect of variations in the magnitude of duplication externalities on model dynamics. We scrutinize the impact of these particular parameters, and not others, for three reasons. First, based on the literature we infer substantial uncertainty in their values. Second, our bifurcation analysis has uncovered that the model dynamics depend critically on the value of the latter of these two parameters, ν_b, but not so much on other uncertain parameters, such as e.g., η_a, η_b. Finally, given our interest in the labor income share, it makes sense to concentrate on parameters whereby endogenous R&D growth is directly affected by labor flows.

Table 11 looks at the consequences of varying the ν's in terms of the implied pace of convergence to the BGP, cycle length, the BGP level of the labor share, and the per-capita growth rate in the decentralized and optimal allocation. We cover symmetric and asymmetric variations in the duplication externalities parameters. These variations around the baseline typically lead to dampened cycles, and occasionally to monotonic convergence. The final column takes all parameters as given, including the particular ν pairings, and varies γ and ρ separately until a Hopf bifurcation is identified.; in

5.5.1 Symmetric Duplication

Based on our numerical results, we find that cycle length, **L**, for both allocations is increasing with the magnitude of duplication externalities (i.e., it is decreasing with $\nu_a = \nu_b$):

$$\left.\frac{\partial \mathbf{L}}{\partial \nu_a}\right|_{\nu_a=\nu_b} < 0.$$

In our baseline case ($\nu_a = \nu_b = 0.75$, in bold) this implies a cycle length of 52-76 years (DA, OA respectively). At the extremes, however, this can change to, e.g., over 150 years or around 30 years (the dominant frequency of medium-term oscillations present in the US data). The intuition behind this result is the following. As duplication externalities fall (the ν values rise), the return from labor flows into each of the R&D sectors becomes higher and the gestation period for new ideas to "come on-stream" is accordingly reduced; thus cycle length shortens.

At $\nu_a = \nu_b = 0.75$ limit cycles arise if the agent's consumption smoothing motive is less strong than in the baseline ($\gamma = 0.94$ vs. 1.75) or if the society becomes more patient ($\rho = 0.006$ vs. 0.02). Again, this makes sense. If the representative household gives a high weight to future generations (low ρ), it is willing to invest substantial resources in physical capital as well as both types of R&D. R&D incurs two types of costs, however: (a) of adjusting the level of labor-augmenting technology a, and (b) of obsolescence of capital-augmenting technologies, with opposing impacts on the labor share. Both impacts also propagate via the mutual R&D spillovers and the "lab equipment" term. If, ultimately, the consumption-smoothing motive is weak enough (low γ), these destabilizing effects are not countered by lowering R&D employment or savings, which leads to endogenous cycles.

Finally, note that the (socially optimal) labor share and per-capita growth rate are increasing in duplication parameters:

$$\left.\frac{\partial(1-\pi^{OA})}{\partial \nu_a}\right|_{\nu_a=\nu_b}, \left.\frac{\partial g^{OA}}{\partial \nu_a}\right|_{\nu_a=\nu_b} > 0.$$

Regarding the labor share, in line with equation (71) we observe that under gross complements, if R&D efficiency increases, inducing labor flows from final goods production towards R&D, this increases the labor share. Naturally, when both R&D technologies become more productive, the balanced growth rate rises as well.

5.5.2 Asymmetric Duplication

The asymmetric case bears some similarities with the symmetric one. There is an increasing profile in the labor share: ceteris paribus, as ν_b increases, the ratio $\dot{\lambda}_a/\dot{\lambda}_b$ rises. Given gross complements, relative improvements in capital-augmenting technical progress are "labor-biased".[38]

As before, with weak duplication externalities in labor-augmenting R&D (large ν_a), there is no limit cycle for any corresponding capital-augmenting value. Again this is intuitive. The balanced growth path is driven by labor-augmenting technologies alone. The more smoothly labor-augmenting ideas are being produced, the closer at any point is the economy to its balanced growth path, and thus less amenable to limit cycles. A corollary of this can be seen when $\nu_b/\nu_a > 1$ (representing the case of "labor-biased duplication", when duplication externalities are stronger in labor-augmenting R&D). Here the economy is far away from balanced technical growth in the sense that the capital-augmenting R&D sector is less constrained by duplication than its labor equivalent. The possibility for waves of innovation and thus excessively fast replacement and obsolence of existing ideas is then more likely to produce exaggerated cycles. In such cases, the consumption smoothing preferences consistent with a limit cycle are still below the baseline value but now closer to and marginally above log preferences.

[38] I.e., raising labor's relative marginal product for given factor proportions.

Table 11: Dynamic Model Properties: Duplication Externalities

Baseline Calibration With ...	$\frac{\nu_b}{\nu_a}$	Pace of Convergence (% Per Annum)		Cycle Length (Years)		$1-\pi^{\dagger}_{OA}$	$g^{\dagger}_{OA}$	Conditional on ν_a, ν_b, the Bifurcation Point For‡	
		DA	OA	DA	OA			γ	ρ
				Symmetric Duplication					
$\nu_a = \nu_b = 0.1$	1	4.84%	3.54%	Monotonic	Monotonic	0.7230	0.0188	0.9237	0.0059
$\nu_a = \nu_b = 0.5$	1	5.58%	4.30%	97.6280	153.7304	0.7635	0.0244	0.9534	0.0064
° a = ° b = **0.75**	1	6.30%	4.20%	52.6563	76.6629	0.7854	0.0339	0.9430	0.0062
$\nu_a = \nu_b = 0.9$	1	9.44%	2.45%	29.4544	42.3207	0.7944	0.0417	No Hopf	No Hopf
				Asymmetric Duplication					
$\nu_a = 0.9, \nu_b = 0.1$	0.1	4.32%	3.88%	Monotonic	161.5625	0.6855	0.0288	No Hopf	No Hopf
$\nu_a = 0.9, \nu_b = 0.5$	0.6	9.93%	3.82%	67.3593	83.8627	0.7345	0.0298	No Hopf	No Hopf
$\nu_a = 0.9, \nu_b = 0.9$	1	9.44%	2.45%	29.4544	42.3207	0.7944	0.0417	No Hopf	No Hopf
$\nu_a = 0.5, \nu_b = 0.9$	1.8	4.53%	5.70%	54.5507	105.0562	0.8249	0.0376	1.0794	0.0086
$\nu_a = 0.1, \nu_b = 0.9$	9.0	4.05%	3.57%	72.3116	Monotonic	0.8425	0.0420	1.0864	0.0090

Note: † The labor share and the per-capita growth rate at the BGP in the decentralized allocation (DA) allocation are exactly matched to the long-run US averages (0.6739, 0.0171 respectively) for each parametrization, and thus are not shown. ‡ following a BGP-preserving sensitivity analysis exercise akin to Figures 5–6. "Monotonic" indicates monotonic convergence to the steady state; otherwise there are dampened oscillations along the convergence path towards the steady state. "No Hopf" indicates that for given ν_a, ν_b, limit cycles cannot be obtained for any γ or ρ. See also notes to Table 10.

Note finally that although our study does not intend to match the frequency of medium-to-long swings exactly, we still view the R&D-based endogenous growth model with CES production as a viable explanation for the hump-shaped trend of the labor share observed in the US throughout the twentieth century. In fact, if one argued that in our 83-year time series of the US labor share, we observe only a half of a full long-run swing, then the model could match that exactly if the imaginary parts of stable roots were around 0.04 which could be obtained e.g., for lower duplication parameters ν_a and ν_b, a higher lab equipment term η_b or lower η_a, etc.

6 Conclusions

The contribution of the current article to the literature has been (i) to document that the observed labor's share of GDP exhibits substantial medium-run swings and volatilities suggestive of a long cycle, and (ii) to provide a theoretical assessment of the extent to which a calibrated endogenous growth model can account for these regularities.

To our knowledge, this is the first paper to emphasize medium-term swings of the labor share, focusing on their technological explanation, and allowing the possibility that these swings be driven by endogenous, stable limit cycles. Our workhorse model has been set up as a micro-founded endogenous growth model with aggregate CES technology and factor-augmenting technical change. We computed both the decentralized and socially optimal solution and analyzed its dynamics. Having calibrated the parameters of our model based on US data, we carried out a sequence of numerical exercises allowing us to:

- confirm that the interplay between endogenous arrivals of capital- and labor-augmenting technologies leads to oscillatory convergence to the long-run growth path as well as stable limit cycles via Hopf bifurcations, and
- assess the magnitude of departures of the decentralized allocation from the first best.

More specifically, our theoretical model has the following implications.

- The model provides an explanation for long swings in the labor share: it implies oscillatory dynamics of factor shares along the convergence path to the BGP.
- The model delivers plausible implications regarding the co-movement of other variables along the long labor share swings.
- Under certain empirically plausible parametrizations, the model gives rise to Hopf bifurcations, indicating the emergence of limit cycles in the labor share.
- Oscillatory behavior of the labor share is due to the interplay between arrivals of capital-augmenting developments (subject to gradual depreciation) and changes in the pace of (continued) labor-augmenting technical change.

Our framework is also suggestive of future work. Consider the following two points. Typically in endogenous growth models a research subsidy is recommended to align the decentralized allocation with the optimal one. In our framework it may be worth investigating whether such a subsidy would also have cyclical characteristics, or whether such a subsidy could be feasibly created. We have also held back from any positive or welfare-based statements. The model laid out here is insufficient for such analysis. But one could envisage analysis aimed at defining whether the cycles (be they convergent or sustained) are or are not welfare enhancing, and thus whether governments intervention is warranted. We leave these for future work.

References

ACEMOGLU, D. (2002): "Technical Change, Inequality, and the Labor Market," *Journal of Economic Literature*, 40(1), 7–72.

ACEMOGLU, D. (2003): "Labor-and Capital-Augmenting Technical Change," *Journal of the European Economic Association*, 1, 1–37.

ACEMOGLU, D., AND V. GUERRIERI (2008): "Capital Deepening and Nonbalanced Economic Growth," *Journal of Political Economy*, 116(3), 467–498.

ARROW, K., H. B. CHENERY, B. S. MINHAS, AND R. M. SOLOW (1961): "Capital-Labor Substitution and Economic Efficiency," *Review of Economics and Statistics*, 43, 225–250.

AUTOR, D. H., F. LEVY, AND R. J. MURNANE (2003): "The Skill Content Of Recent Technological Change: An Empirical Exploration," *Quarterly Journal of Economics*, 118(4), 1279–1333.

BAI, J., AND P. PERRON (2003): "Computation and analysis of multiple structural change models," *Journal of Applied Econometrics*, 18(1), 1–22.

BARRO, R. J., AND X. X. SALA-I-MARTIN (2003): *Economic Growth*. MIT Press.

BASU, S., AND J. FERNALD (1997): "Returns to Scale in U.S. Manufacturing: Estimates and Implications," *Journal of Political Economy*, 105(2), 249–283.

BENHABIB, J., AND K. NISHIMURA (1979): "The Hopf bifurcation and the existence and stability of closed orbits in multiscctor models of optimal economic growth," *Journal of Economic Theory*, 21, 421–444.

BENTOLILA, S., AND G. SAINT-PAUL (2003): "Explaining Movements in the Labor Share," *Contributions to Macroeconomics*, 3.

BERNDT, E. R. (1976): "Reconciling Alternative Estimates of the Elasticity of Substitution," *Review of Economics and Statistics*, 58, 59–68.

BLOOM, N., M. SCHANKERMAN, AND J. V. REENEN (2013): "Identifying Technology Spillovers and Product Market Rivalry," *Econometrica*, 81, 1347–1393.

BOLDRIN, M., AND M. WOODFORD (1990): "Equilibrium models displaying endogenous fluctuations and chaos : A survey," *Journal of Monetary Economics*, 25(2), 189–222.

BUERA, F. J., AND J. P. KABOSKI (2012): "The Rise of the Service Economy," *American Economic Review*, 102(6), 2540–69.

CASELLI, F. (2005): "Accounting for Cross-Country Income Differences," in *Handbook of Economic Growth*, ed. by P. Aghion, and S. Durlauf. North-Holland.

CHIRINKO, R. S. (2008): "σ: The Long and Short of It," *Journal of Macroeconomics*, 30, 671–686.

CHRISTIANO, L. J., AND T. J. FITZGERALD (2003): "The Band Pass Filter," *International Economic Review*, 44(2), 435–465.

COMIN, D., AND M. GERTLER (2006): "Medium-Term Business Cycles," *American Economic Review*, 96(3), 523–551.

CORREA-LÓPEZ, M., AND B. DE BLAS (2012): "International Transmission of Medium-Term Technology Cycles: Evidence from Spain as a Recipient Country," *The B.E. Journal of Macroeconomics*, 12(1), 1–52.

DE HAAN, L., AND A. FERREIRA (2006): *Extreme Value Theory*. Springer.

DE LA GRANDVILLE, O. (1989): "In Quest of the Slutsky Diamond," *American Economic Review*, 79, 468–481.

——— (2009): *Economic Growth: A Unified Approach*. Cambridge University Press.

DE SERRES, A., S. SCARPETTA, AND C. DE LA MAISONNEUVE (2002): "Sectoral Shifts in Europe and the United States: How They Affect Aggregate Labour Shares and the Properties of Wage Equations," Working paper, OECD Economics Department WP 326.

DOCKNER, E. J. (1985): "Local stability analysis in optimal control problems with two state variables," in *Optimal Control Theory and Economic Analysis Vol. 2*, ed. by G. Feichtinger, pp. 89–103. Amsterdam: North-Holland.

ELSBY, M. W., B. HOBIJN, AND A. SAHIN (2013): "The decline of the U.S. labor share," Working Paper Series 2013-27, Federal Reserve Bank of San Francisco.

FARIA, J., AND J. ANDRADE (1998): "Investment, credit, and endogenous cycles," *Journal of Economics*, 67(2), 135–143.

FEICHTINGER, G. (1992): "Limit Cycles in Dynamic Economic Systems," *Annals of Operations Research*, 37, 313–344.

GOLLIN, D. (2002): "Getting Income Shares Right," *Journal of Political Economy*, 110, 458–474.

GOMME, P., AND P. RUPERT (2007): "Theory, measurement and calibration of macroeconomic models," *Journal of Monetary Economics*, 54(2), 460–497.

GOODWIN, R. (1951): "The non-linear Accelerator and the persistence of business cycles," *Econometrica*, 19, 1–17.

GROWIEC, J. (2008): "Production Functions and Distributions of Unit Factor Productivities: Uncovering the Link," *Economics Letters*, 101, 87–90.

——— (2013): "A Microfoundation for Normalized CES Production Functions with Factor-Augmenting Technical Change," *Journal of Economic Dynamics and Control*, 37, 2336–2350.

HAMILTON, J. D. (1989): "A New Approach to the Economic Analysis of Nonstationary Time Series and the Business Cycle," *Econometrica*, 57(2), 357–84.

HANSEN, B. E. (1997): "Approximate Asymptotic P Values for Structural-Change Tests," *Journal of Business & Economic Statistics*, 15(1), 60–67.

HOUTHAKKER, H. S. (1955): "The Pareto Distribution and the Cobb-Douglas Production Function in Activity Analysis," *Review of Economic Studies*, 23, 27–31.

JONES, C. I. (1995): "R&D-Based Models of Economic Growth," *Journal of Political Economy*, 103, 759–84.

——— (1999): "Growth: With or Without Scale Effects?," *American Economic Review*, 89(2), 139–144.

——— (2002): "Sources of U.S. Economic Growth in a World of Ideas," *American Economic Review*, 92, 220–239.

——— (2005a): "Growth and Ideas," in *Handbook of Economic Growth*, ed. by P. Aghion, and S. Durlauf. North-Holland.

——— (2005b): "The Shape of Production Functions and the Direction of Technical Change," *Quarterly Journal of Economics*, 120, 517–549.

JONES, C. I., AND J. C. WILLIAMS (2000): "Too Much of a Good Thing? The Economics of Investment in R&D," *Journal of Economic Growth*, 5(1), 65–85.

KALDOR, N. (1940): "A model of the trade cycle," *Economic Journal*, 50, 78–92.

——— (1961): "Capital Accumulation and Economic Growth," in *The Theory of Capital*, ed. by L. A. Lutz, and D. C. Hague. Palgrave Macmillan.

KARABARBOUNIS, L., AND B. NEIMAN (2014): "The Global Decline of the Labor Share," *The Quarterly Journal of Economics*, 129(1), 61–103.

KLUMP, R., AND O. DE LA GRANDVILLE (2000): "Economic Growth and the Elasticity of Substitution: Two Theorems and Some Suggestions," *American Economic Review*, 90, 282–291.

KLUMP, R., P. MCADAM, AND A. WILLMAN (2007): "Factor Substitution and Factor Augmenting Technical Progress in the US: A Normalized Supply-Side System Approach," *Review of Economics and Statistics*, 89, 183–192.

——— (2012): "Normalization in CES Production Functions: Theory and Empirics," *Journal of Economic Surveys*, 26, 769–799.

KONGSAMUT, P., S. REBELO, AND D. XIE (2001): "Beyond Balanced Growth," *Review of Economic Studies*, 68(4), 869–82.

KREMER, M. (1993): "The O-Ring Theory of Economic Development," *Quarterly Journal of Economics*, 108, 551–575.

KYYRÄ, T., AND M. MALIRANTA (2008): "The Micro-Level Dynamics of Declining Labour Share: Lessons from the Finnish Great Leap," *Industrial and Corporate Change*, 17, 1146–1172.

LEÓN-LEDESMA, M. A., P. MCADAM, AND A. WILLMAN (2010): "Identifying the Elasticity of Substitution with Biased Technical Change," *American Economic Review*, 100, 1330–1357.

LEON-LEDESMA, M. A., AND M. SATCHI (2011): "The Choice of CES Production Techniques and Balanced Growth," Studies in Economics 1113, Department of Economics, University of Kent.

LI, C.-W. (2000): "Endogenous vs. Semi-endogenous Growth in a Two-R&D-Sector Model," *Economic Journal*, 110(462), C109–22.

MCADAM, P., AND A. WILLMAN (2004): "Supply, Factor Shares and Inflation Persistence: Re-Examining Euro-Area New-Keynesian Phillips Curves," *Oxford Bulletin of Economics and Statistics*, 66(9), 637–670.

NGAI, L. R., AND C. A. PISSARIDES (2007): "Structural Change in a Multisector Model of Growth," *American Economic Review*, 97(1), 429–443.

OECD/EUROSTAT (2005): *Oslo Manual*. OECD Publishing.

PERRON, P. (1989): "The great crash, the oil price shock, and the unit root hypothesis.," *Econometrica*, 57, 361–401.

PESSOA, A. (2005): "Ideas' driven growth: the OECD evidence," *Portuguese Economic Journal*, 4(1), 46–67.

PIKETTY, T. (2014): *Capital in the Twenty-First Century*. Harvard University Press, Translated from French by Arthur Goldhammer.

ROMER, P. M. (1990): "Endogenous Technological Change," *Journal of Political Economy*, 98, S71–S102.

RYDER, H. E., AND G. M. HEAL (1973): "Optimum Growth with Intertemporally Dependent Preferences," *Review of Economic Studies*, 40(1), 1–33.

SOLOW, R. M. (1956): "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics*, 70, 65–94.

——— (1960): "Investment and Technical Progress," in *Mathematical Methods in the Social Sciences*, ed. by K. Arrow, S. Karlin, and P. Suppes, pp. 89–104. Stanford University Press.

STOKEY, N. L. (1995): "R&D and Economic Growth," *Review of Economic Studies*, 62(3), 469–89.

UZAWA, H. (1961): "Neutral Inventions and the Stability of Growth Equilibrium," *Review of Eco-*

nomic Studies, 28, 117–124.
YOUNG, A. (2004): "Labor's Share Fluctuations, Biased Technical Change, and the Business Cycle," *Review of Economic Dynamics*, 7(4), 916–931.

A Data Construction

A.1 Labor Share

The broadly used approach in measuring the labor share is simply dividing Compensation of Employees (CE) by the GDP. But that does not take into consideration the self-employees income. Unfortunately, self-employees labor income is published with the capital income. Since Gollin (2002) a few adjustments have been proposed. We incorporate one of the most detailed way in the measuring labor share suggested by Gomme and Rupert (2007) and which takes into consideration the unknown (self-employees) income. The starting point is the assumption that proportion of the unknown labor (capital) income to the total unknown income is the same as the ratio of known labor (capital) to the known income of both share. The unknown income (AI) is the sum of Proprietor's Income (PI), Business Current Transfer Payments ($BCTP$), Statistical Discrepancy ($SDis$) and Taxes on Production (Tax) reduced by Subsidies (Sub) ($AI = PI + Tax - Sub + BCTP + SDis$). On the other hand, known capital income (UCI) consists of Rental Income (RI), Current Surplus of Government Enterprises (GE), Net Interests (NI) and Corporate Profits (CP). Including UCI to Compensation of Employees (CE) and consumption of fixed capital (DEP) we derive total unambiguous income (UI) and can calculate the portion of UCI to UI: $\kappa = \dfrac{UCI + DEP}{UI}$. Having κ it is easy to obtain ambiguous capital income (ACI) which equals $AC \cdot \kappa$. Finally, we derive labor share income as one minus capital income share:

$$LS_t = 1 - \frac{UCI + DEP + ACI}{GDP} = 1 - \kappa \tag{A.1}$$

GDP and Consumption of fixed capital (DEP) are taken from NIPA*[Table 1.7.5]* and the rest series are taken from NIPA *[Table 1.12]*.

A.2 Macroeconomic Variables

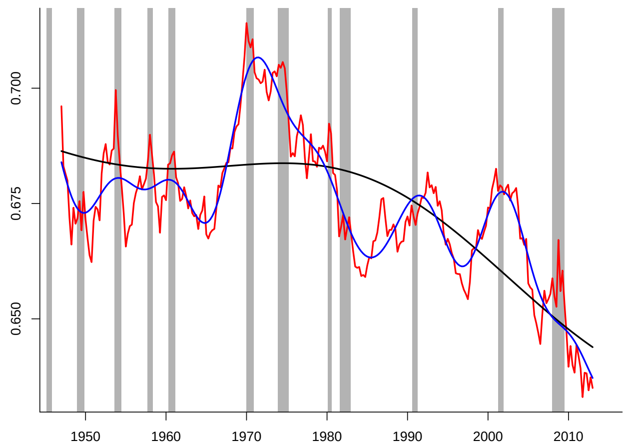
GDP - Gross Domestic Product in billions of chained (2005) dollars, BEA NIPA Table 1.6.
Labor productivity (LP_t) - Labor Productivity in nonfarm business sector, index (2009=100), BLS Series No. PRS85006093.
Consumption (C_t) - Personal consumption expenditures in billions of chained (2005) dollars, BEA NIPA Table 1.6.
Investment (I_t) - Gross private domestic investment in billions of chained (2005) dollars, BEA NIPA Table 1.6.
Government expenditures (G_t) - Government consumption expenditures and gross investment in

billions of chained (2005) dollars, BEA NIPA Table 1.6.
Consumption to product ratio (C_t/Y_t) - Personal consumption expenditures (in billions of current dollars) divided by Gross Domestic Product (in billions of current dollars), both series from BEA NIPA Table 1.5.
Investment to product ratio (I_t/Y_t) - Gross private domestic investment (in billions of current dollars) divided by Gross Domestic Product (in billions of current dollars), both series from BEA NIPA Table 1.5.
Consumption of durables goods to product ratio (C_t^{DG}/Y_t) - Personal consumption expenditures on durables goods (in billions of current dollars) divided by Gross Domestic Product (in billions of current dollars), both series from BEA NIPA Table 1.5.
Consumption of non-durables goods to product ratio (C_t^{NDG}/Y_t) - Personal consumption expenditures on non-durables goods (in billions of current dollars) divided by Gross Domestic Product (in billions of current dollars), both series from BEA NIPA Table 1.5.
R&D expenditures (RD_t) – Research and development expenditures in constant millions of dollars, series taken from National Science Foundation.
The share of the R&D expenditures in Gross Domestic Product (RD_t/GDP_t) is the ratio of total spending on research and development sector divided by GDP, both series in current millions of dollars, taken from National Science Foundation and BEA NIPA Table 1.5, respectively.
Employment (E_t) - Employment in non-farm business sector, index (2009=100), BLS Series No. PRS85006013.
Hours (H_t) - Hours in non-farm business sector, index, BLS Series No. PRS85006033.
Consumption to private physical capital stock (C_t/K_t) - ratio of consumption expenditures in current billions of dollars to private fixed assets stock also in current billions of dollars, series taken from BEA NIPA Table 1.5 and BEA Fixed Assets Table 1.1 respectively.
Private capital stock to product ($K_t^{PRIVATE}/GDP_t$) - private fixed assets stock in current billions of dollars to Gross Domestic Product, series taken from BEA Fixed Assets Table 1.1 and BEA NIPA Table 1.5 respectively, time span.
Aggregate hours of unskilled workers ($hours_t^U$) - the aggregated hours in all sectors for all employees with education not higher than college, raw data taken from World KLEMS database.
Aggregate hours of skilled workers ($hours_t^U$) - the aggregated hours in all sectors for all employees with education level equivalent at least some college, raw data taken from World KLEMS database.
The ratio of skilled to unskilled hours ($hours_t^S/hours_t^U$) - calculated as a ratio of series described above.
Skill premium (w_t^S/w_t^U) – calculated in the following way. First, we calculate total compensation of employees for both skilled and unskilled workers. The disaggregation is determined by the education achievement as above. Second, we calculate unit compensation per hour for each group. That constructed unit wages are used in final calculation.

B Additional Empirical Results

B.1 Additional Figures and Tables

Figure B.1: The Quarterly Labor Share, Its Medium-Term Component and Long-Term Trend



Note: the red, blue and black lines represents for raw, medium-term component and long-run trend, respectively.

Figure B.2: The Output-to-Capital Ratio

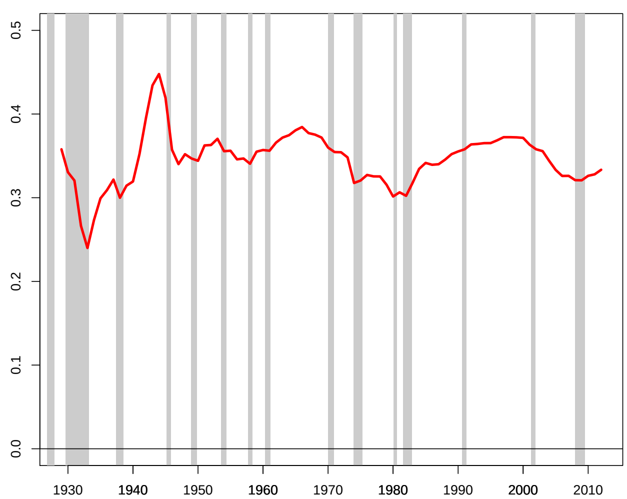
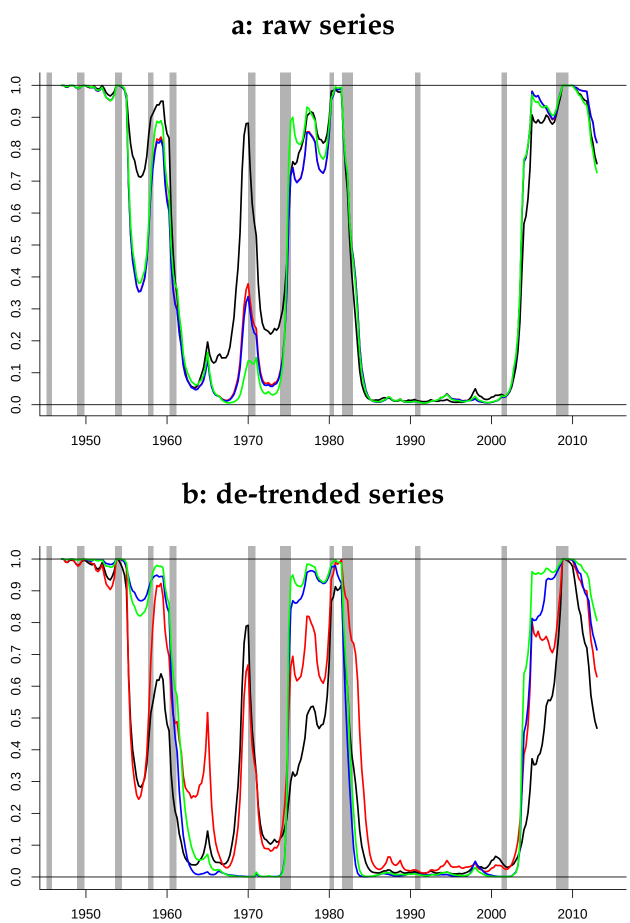


Figure B.3: Filtered Probability of the High-Variance Regime in the Markov Switching Model



Note: the black, red, blue and green lines represents smoothed probability of high-variance regime for the specifications (1), (2), (3) and (4), respectively.

Table B.1: Labor Share: Summary Statistics

	annual	quarterly
Mean	0.674	0.674
Max	0.711	0.714
Min	0.621	0.633
Std. Dev.	0.017	0.015
Skewness	−0.369	−0.106
Kurtosis	3.947	3.358
Normality	[0.080]	[0.385]
Obs.	84	265

Note: Normality test is Jarque-Bera.

Table B.2: Labor Share: Stationarity Tests

	annual				quarterly			
	intercept		trend & intercept		intercept		trend & intercept	
		CV 5%		CV 5%		CV 5%		CV 5%
ADF	[0.004]	–	[0.004]	–	[0.493]	–	[0.493]	–
ERS DF-GLS	−1.480	−1.940	−2.066	−3.088	−0.353	−1.942	−2.24596	−2.9172
PP	[0.036]	–	[0.022]	–	[0.428]	–	[0.390]	–
KPSS	0.339	0.463	0.245	0.146	0.931	0.463	0.247	0.146
ERS	7.530	3.070	15.913	5.666	12.790	3.199	9.076	5.647
Ng-Perron								
MZa	−4.546	−8.100	−10.444	−17.300	−0.985	−8.100	−10.506	−17.300
MZb	−1.507	−1.980	−2.145	−2.910	−0.346	−1.980	−2.204	−2.910
MSB	0.331	0.233	0.205	0.168	0.351	0.233	0.210	0.168
MPT	5.391	3.170	9.392	5.480	11.312	3.170	9.107	5.480

Note: Regressions performed in levels with sequentially an intercept, then intercept plus linear trend. Squared brackets indicate probability values and CV5% denotes 5% critical value of the test. ADF-Augmented Dickey Fuller; ERS DF-GLS=Elliott-Rothenberg-Stock (1996), Dickey-Fuller GLS; PP=Philips-Perron; KPSS=Kwiatkowski-Phillips-Schmidt-Shin (1992); ERS=Elliott-Rothenberg-Stock (1996) point-optimal unit root; multiple Ng-Perron (2001) tests. Descriptions of these tests can be readily found in econometrics textbooks. The null in each case is that the series has a unit root (except for the KPSS test which has stationarity as the null). In each case the number of lags in the stationarity equation is determined by Schwartz Information criteria. In the Philips-Perron and KPSS methods, we use the Bartlett Kernel as the spectral estimation method and Newey-West bandwidth selection.

B.2 Labor Share vs. the Capital–Output Ratio

B.2.1 A Simple Auto-regressive Distributed Lag Specification

To investigate the relationship between the logarithm of the capital-to-output ratio ($\ln(y/k)_t$) and the log labor share ($\ln(ls)_t$), we have estimated a simple ARDL model:

$$\ln(ls)_t = \mu + \sum_{i=1}^{p} \rho_i \ln(ls)_{t-i} + \sum_{i=0}^{q} \beta_q \ln(y/k)_{t-i} + \varepsilon_t \tag{A.2}$$

Our ARDL model estimates are presented in Table B.3. For the models without the autoregressive part, Newey-West standard errors are used to solve the auto-correlation problem. It turns out that the dynamic relationship between both variables is weak. Even if we include lagged output-to-capital ratio, the estimate of β_0 is not robustly different from zero. In addition, including an autoregressive part (to reduce omitted-variable bias) leads to a reduction in β_i estimates. This points to the conclusion that capital deepening does not appear as a powerful explanation for labor share movements.

Table B.3: The Labor Share vs. the Capital–Output Ratio: ARDL Regression

	(1)	(2)	(3)	(4)
$\hat{\mu}$	0.635***	0.606***	0.096***	0.108***
$\hat{\beta}_0$	0.112	0.057	0.069**	-0.042
$\hat{\beta}_1$		0.142*		0.128**
$\hat{\rho}_1$			0.823***	0.796***

B.2.2 Simple Structural VAR

We have also scrutinized the relationship between the output-to-capital ratio and the labor share based on a simple, two-dimensional structural VAR model. In order to avoid the autocorrelation problem, we have included two lags in the reduced form specification. Two SVAR models are considered here: with a short-run restriction (no contemporaneous impact of labor share shocks on the capital-to-output ratio), and a long-run restriction (no long-run impact).

The forecast error variance decomposition for the labor share is presented in Figure B.5. Our main finding is that (at least under this simple specification) short-run volatility of the labor share is driven by shocks to the capital-output-ratio to a very small extent. Our identification implies that at most 25% of the labor share forecast error variance is generated by shocks from $(y/k)_t$.

Figure B.4: IRF for the Labor Share After a Shock in $(y/k)_t$

a: short-run SVAR | **b: SVAR with long-run restrictions**

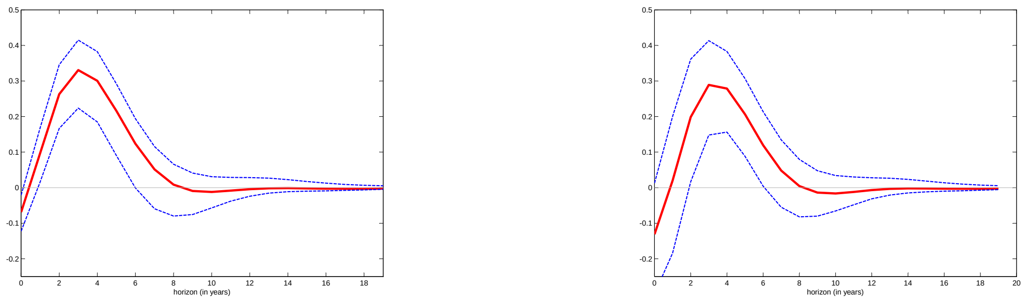
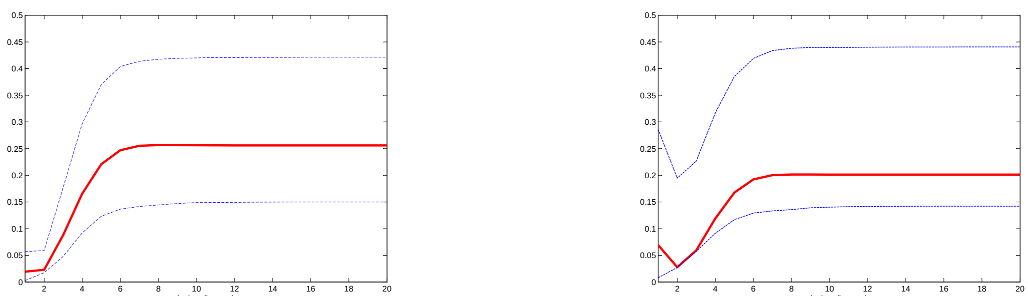


Figure B.5: The Share of y/k_t Shock in Overall Labor Share Forecast Error Variance

a: short-run SVAR | **b: SVAR with long-run restrictions**



C Additional Numerical Results

Figure C.6: Additional Bifurcation Figures

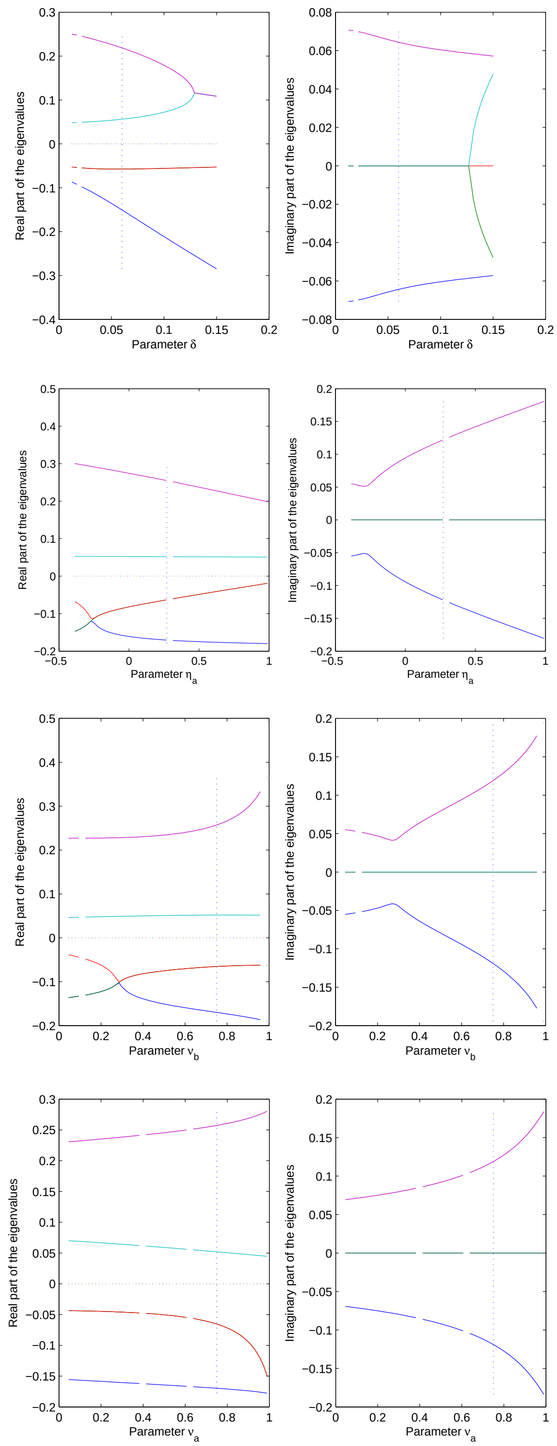


Figure C.7: Non-BGP-Preserving Bifurcation Figures: DA vs. OA

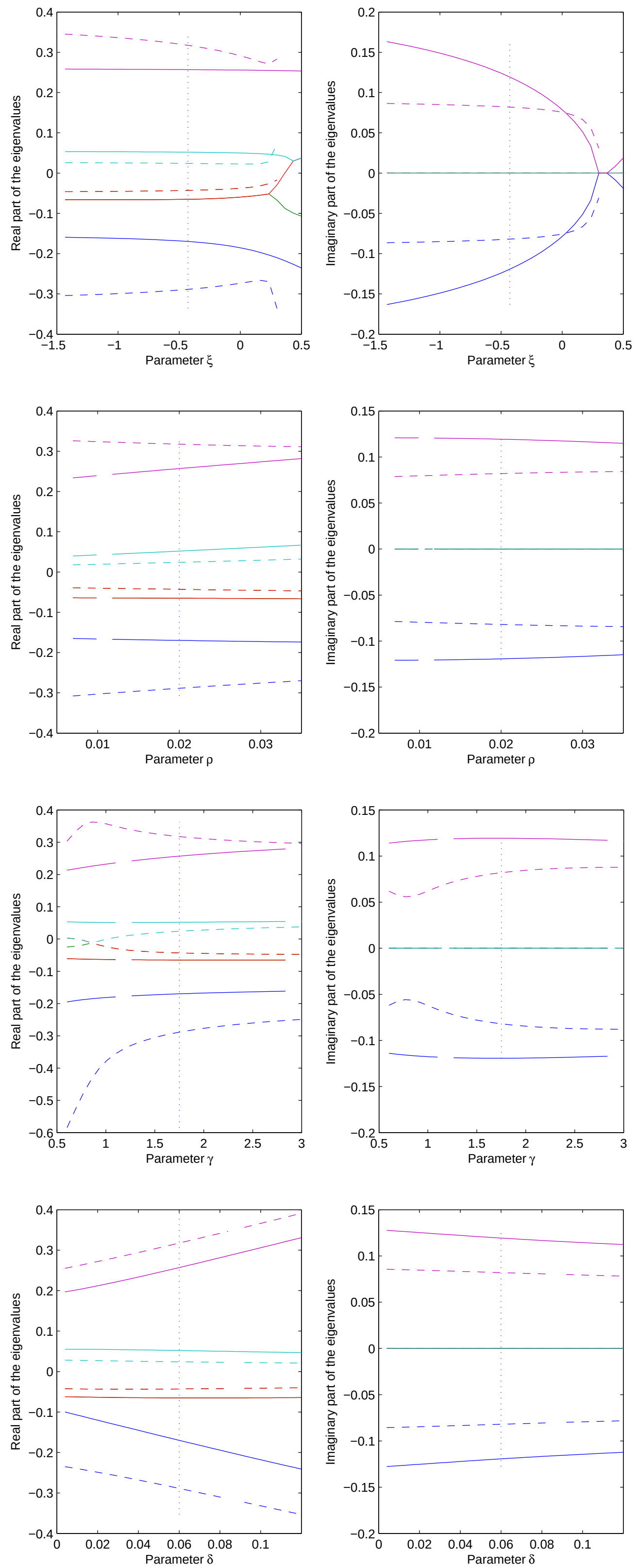


Figure C.8: Non-BGP-Preserving Bifurcation Figures: DA vs. OA (Continued)

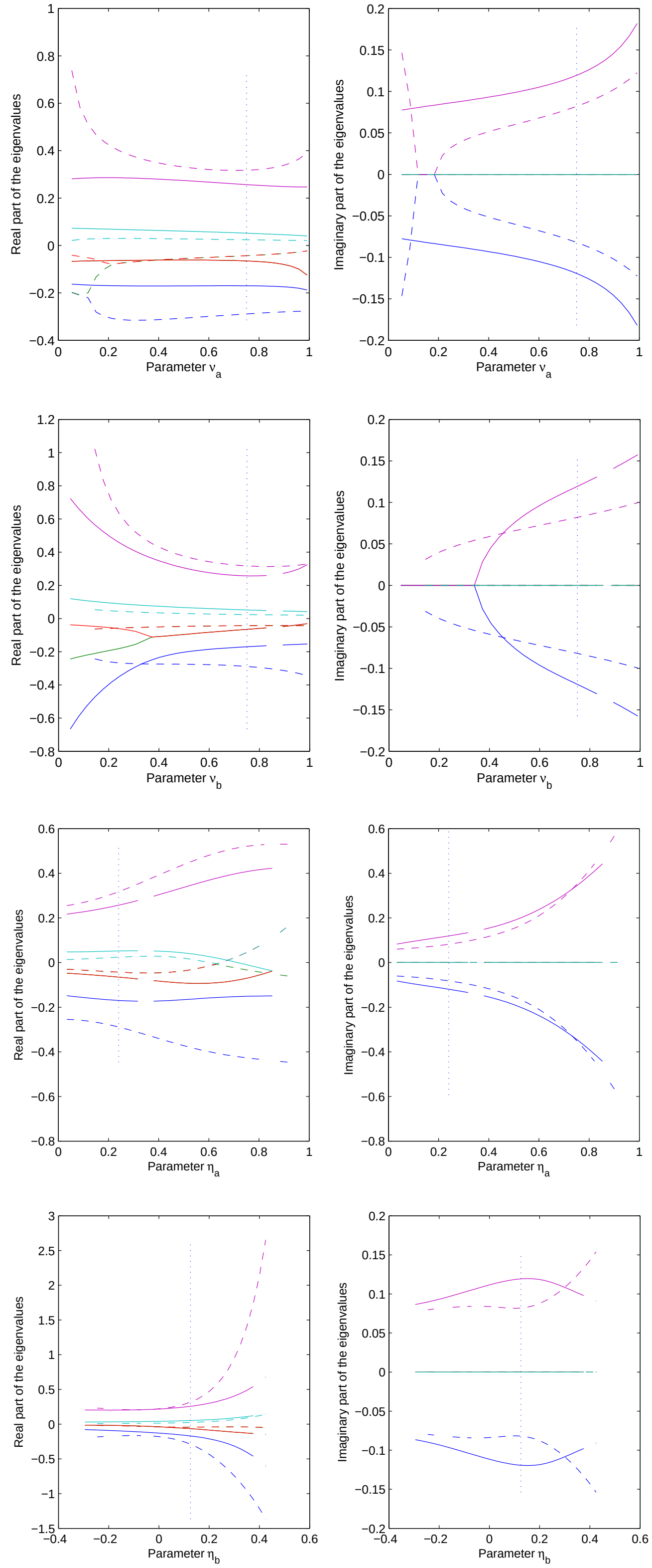


Figure C.9: Comparing Balanced Growth Paths, DA vs. OA. Dependence on the Elasticity of Substitution.

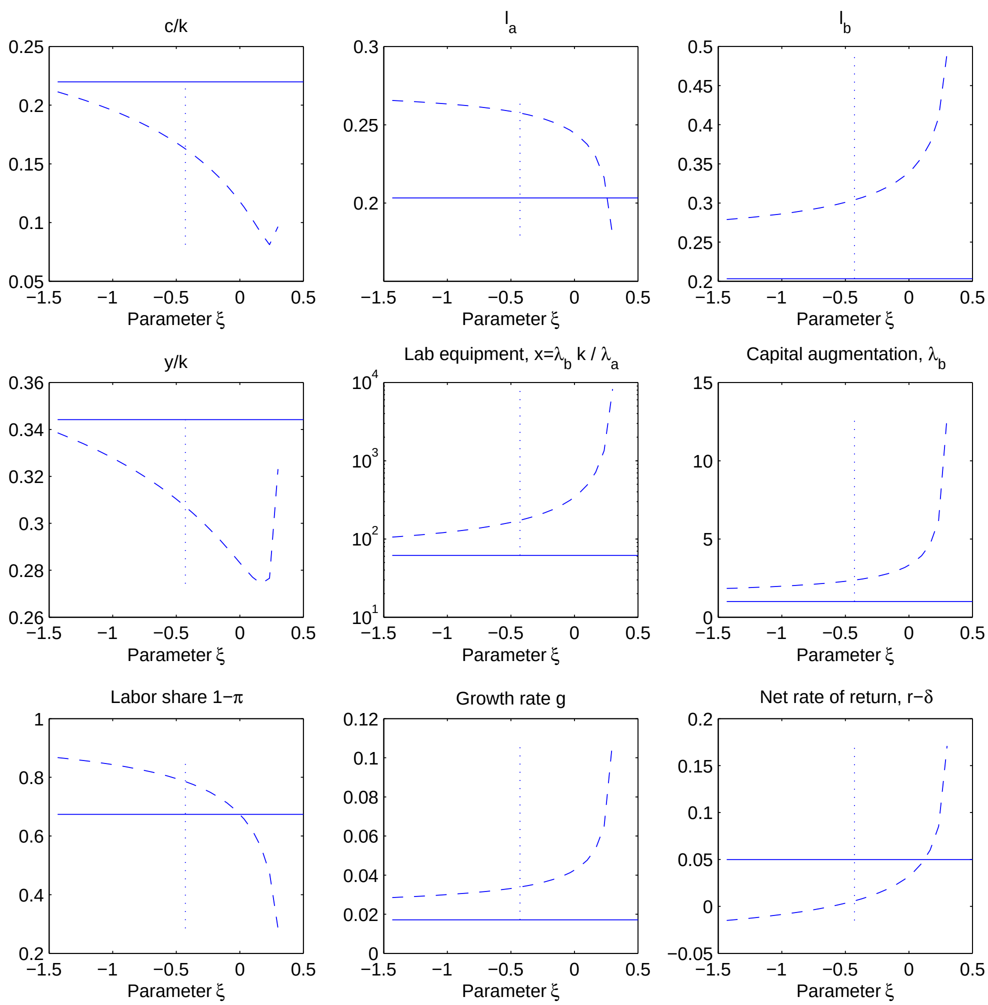


Figure C.10: Comparing Balanced Growth Paths, DA vs. OA. Dependence on the Time Preference.

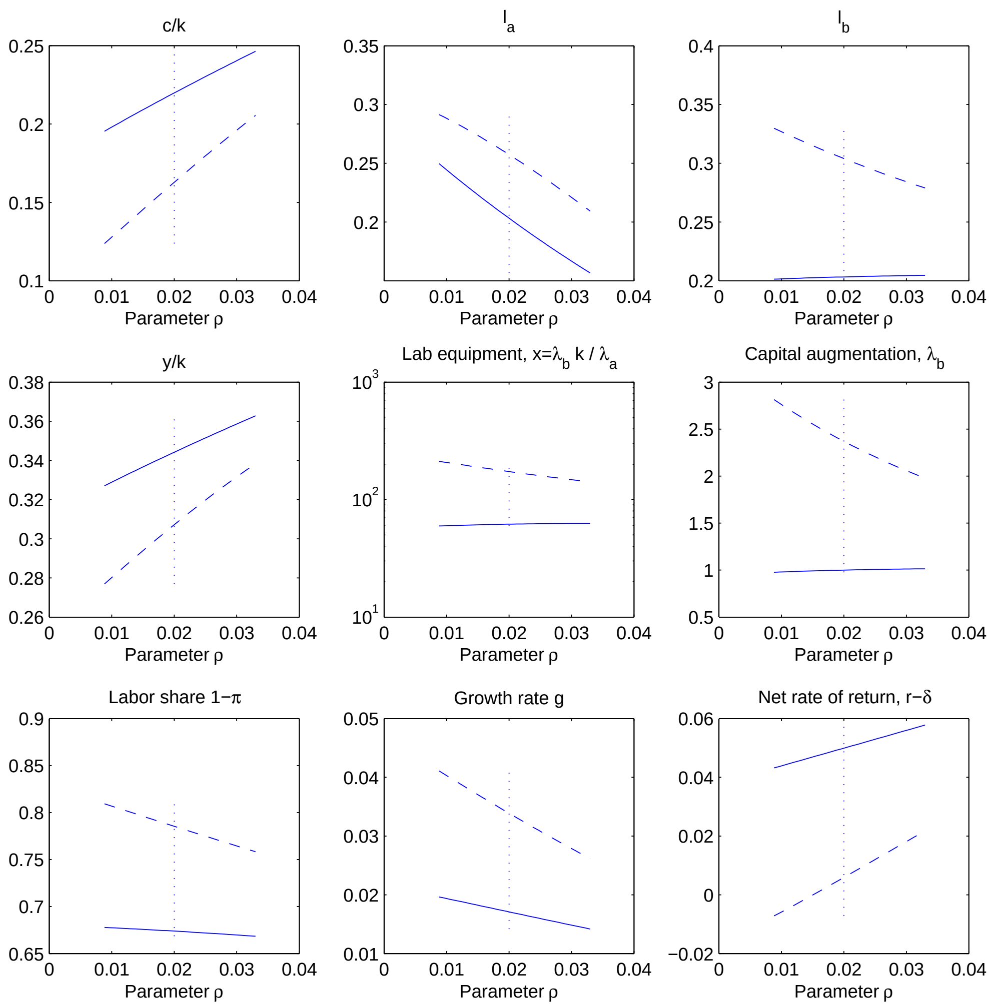


Figure C.11: Comparing Balanced Growth Paths, DA vs. OA. Dependence on the Intertemporal Elasticity of Substitution in Consumption.

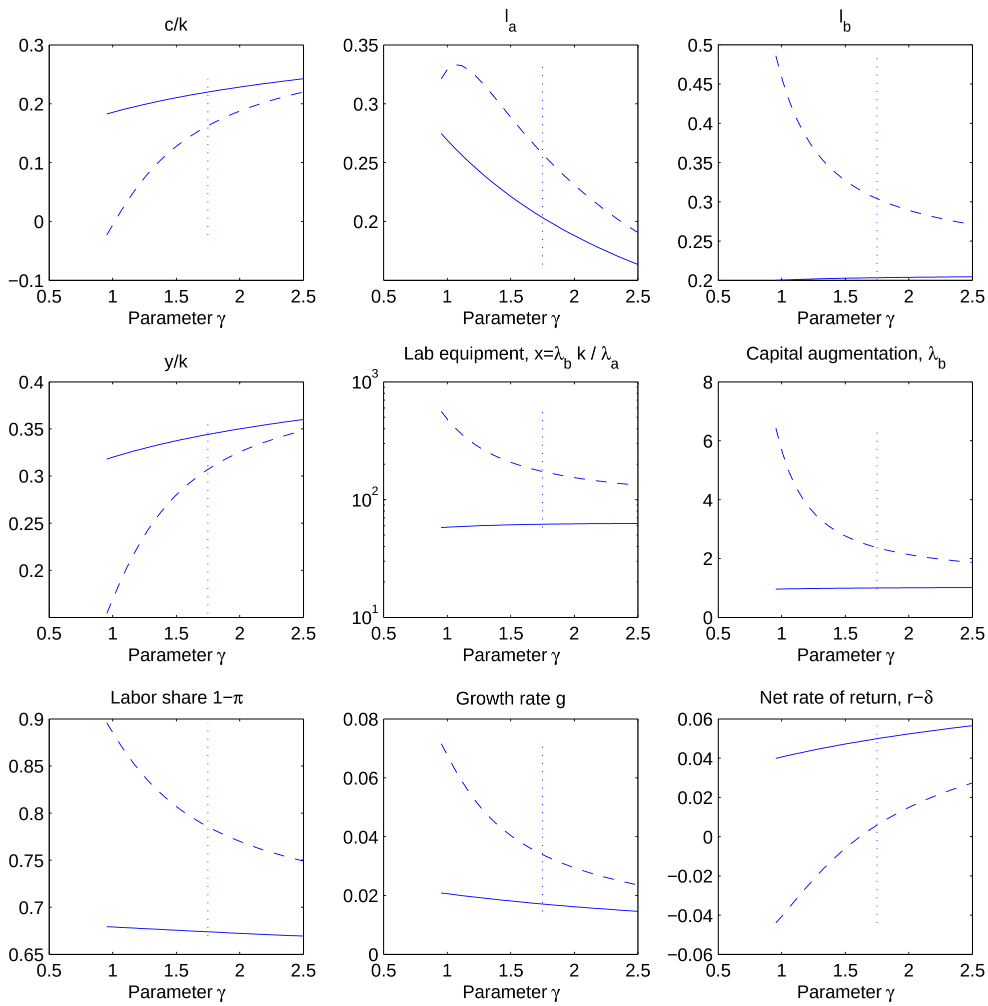


Figure C.12: Comparing Balanced Growth Paths, DA vs. OA. Dependence on ν_b.